POISSON BOUNDARY OF GROUPS ACTING ON $\mathbb{R}$-TREES

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Abstract. We give a geometric description of the Poisson boundaries of certain extensions of free and hyperbolic groups. In particular, we get a full description of the Poisson boundaries of free-by-cyclic groups. We rely upon the description of Poisson boundaries by means of a topological compactification as developed by Kaimanovich. All the groups studied here share the property of admitting a sufficiently complicated action on some $\mathbb{R}$-tree.

Introduction

Let G be a finitely generated group and μ a probability measure on G. A bounded function $f : G \to \mathbb{R}$ is *μ-harmonic* if

$$\forall g \in G\ ,\ f(g) \ =\ \sum_h \mu(h) f(gh)\ .$$

Denote by $H^\infty(G,\mu)$ the Banach space of all bounded μ-harmonic functions on G equipped with the sup norm. In [31, 32], Furstenberg constructed a measured G-space Γ with a μ-stationary measure ν such that a version of the Poisson formula states an isometric isomorphism between $L^\infty(\Gamma,\nu)$ and $H^\infty(G,\mu)$. The space (Γ,ν) is called the *Poisson boundary* of the measured group (G,μ). If the support of μ generates the whole group G, then there are no non-constant bounded μ-harmonic function on G if and only if the Poisson boundary (Γ,ν) is trivial. This is the *Liouville property*.

The Poisson boundary is closely related to some asymptotic properties of the *random walk* associated to the pair (G,μ). The random walk is the Markov chain on G with the transition probabilities $p(x,y) = \mu(x^{-1}y)$. The position x_n of the random walk at time n is obtained from its position x_0 at time 0 by multiplying by independant μ-distributed right increments h_i: $x_n = x_0 h_1 h_2 \cdots h_n$. In [45] (see also [49, 48]), Kaimanovich defines the Poisson boundary as being the space of ergodic components of the time shift T in the space of sample paths $G^{\mathbb{N}}$ of the random walk, i.e. the measurable envelope of the orbit equivalence relation of the shift T. As a consequence, if a.e. path $\mathbf{x} = \{x_n\}$ converges to some (random) limit x_∞ in some compactifiction $\overline{G}$ of G, then the space $\overline{G}$ endowed with the hitting measure λ (i.e. the distribution of x_∞) is a quotient of the Poisson boundary (Γ,ν), and the Poisson formula enables to identify the space $L^\infty(\overline{G},\lambda)$ with some closed subspace of $H^\infty(G,\mu)$.

The Poisson boundary is trivial for all measures on abelian groups [18] and nilpotent groups [24]. In [2], Avez introduced the notion of asymptotic entropy $h(G,\mu)$ of the pair (G,μ), and proved that if μ has finite support and $h(G,\mu) = 0$ then the Poisson boundary is trivial, see [3, 4]. The reciprocal was proved independently by Derriennic [23] and Kaimanovich and Vershik [44]. Particularly, triviality of the Poisson boundary occurs

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if G has subexponential growth (and μ has finite support) [3]. Recently, A.Erschler gave an example of group with subexponential growth and a measure μ infinitely supported with non-trivial Poisson boundary, see [25]. The Liouville property is also related to amenability, namely: a group G is amenable if and only if it admits a measure μ the support of which generates G and such that the Poisson boundary of (G, μ) is trivial, see [44, 66]. Particularly, the Poisson boundary of any non-amenable group (with a non-degenerate measure) is non-trivial. Depending on the measure μ, both triviality and non-triviality can occur on amenable groups G with exponential growth, see [45]. Recently, Bartholdi, Kaimanovich and Nekrashevych used the entropy criterion to prove the amenability of groups generated by finite bounded automata [8].

Beyond the question of triviality, it is natural to wish to fully understand the Poisson boundary. There is a description of the Poisson boundary of the free group (with the uniform measure on the generators) in [24] by means of right-infinite words.. The case of discrete subgroup of $SL(n, \mathbb{R})$ is treated in [32] where the Poisson boundary is related to the space of flags. For random walks on Lie groups, the study and the description of the Poisson boundary started in [31] and was extensively developed in the 70's and the 80's by many authors including Furstenberg [32, 33], Azencott [5], Raugi and Guivarc'h [63, 43, 64], Ledrappier [52, 7]. Some analogies between discrete groups and their continuous counterparts are enlightened, see [32, 52, 7], but some contrast can appear, compare [64] with [44]. The Poisson boundary of some Fuchsian groups has also been described by Series as being the limit set of the group [67]. Kaimanovich and Masur [49] gave a description of the the Poisson boundary of the mapping class group in terms of projective measured foliations. Their description also runs for the Poisson boundary of the braid group, see [27].

The Poisson boundary is closely related to some other well-known compactifications or boundaries. The Martin boundary is involved in the description of *positive* harmonic functions (see [1]). Considered as a measure space with the representing measure of the constant harmonic function $\mathbf{1}$, it is isomorphic to the Poisson boundary [46]. If the group G has infinitely many ends (and if μ is non-degenerate) then the Poisson boundary can be identified with the space of ends (with the hitting measure), see [69, 17, 48]. Furthermore, Kaimanovich obtained an identification of the Poisson boundary for hyperbolic groups with the geometric boundary, see [48]. Concerning the Floyd boundary [30], Karlsson recently proved that if it is not trivial, then it is isomorphic to the Poisson boundary, see [51]. The identification of these various compactifications with the Poisson boundary strongly relies on a general entropic criterion developed by Kaimanovich in [48] (and already used in [45, 44, 49]). This criterion will be our central tool in the present paper.

The groups we deal with in this paper mainly are extensions of free, and more generally hyperbolic groups: what becomes the Poisson boundary of a hyperbolic group G (which is known to be the space of ends of geodesic rays - see [48]) when taking an extension $\mathcal{G}$ of G ? Of course, if it happens that $\mathcal{G}$ itself is a hyperbolic group, then its Poisson boundary may be identified with its hyperbolic boundary. But this answer is not completely satisfactory for us since the link between the Poisson boundaries of G and of $\mathcal{G}$ remains unclear. So far, the only result that we know is the following: if $\mathcal{G}$ is an extension of a free group G and μ a non-degenerate probability measure on $\mathcal{G}$ then the hyperbolic boundary of G is a μ-boundary for $\mathcal{G}$, that is a quotient of the Poisson boundary of $\mathcal{G}$, see [68]. Besides an extension of a hyperbolic group often is not itself hyperbolic (we refer the reader to [9] or [38, 36] for results on this subject). The approach we took to tackle this problem was inspired by the long line of works about free group automorphims starting at the beginning of the nineties, see [12, 11, 34, 35, 55, 56, 19, 21] to cite only a few of them. A common feature to the papers [34, 35, 55, 56, 19, 21] is the introduction of a so-called "α-projectively invariant $\mathbb{F}_n$-tree", that is a $\mathbb{R}$-tree (a geodesic metric space such that any two points are connected by a unique arc, and this arc is a geodesic for the considered metric) with an isometric action of $\mathbb{F}_n$ and a homothety H_α satisfying the fundamental relation: $H_\alpha(w.P) = \alpha(w).H_\alpha(P)$ (if the homothety H_α is an isometry then the $\mathbb{R}$-tree is "α-invariant"). Thanks to this relation, we have in fact an action of the semi-direct product $\mathbb{F}_n \rtimes_\alpha \mathbb{Z}$ on the $\mathbb{R}$-tree. Having in mind the problem of compactifying such a semi-direct product, a major drawback of these $\mathbb{R}$-trees is that the usual adjonction of their Gromov-boundary does not provide us with a compact space. This point is however settled by equipping them with the so-called "observers topology": the union of the completion of the $\mathbb{R}$-tree $\mathcal{T}$ with its Gromov boundary $\partial\mathcal{T}$, equipped with the observers topology, is denoted by $\widehat{\mathcal{T}}$. This weak topology is thoroughly studied in [19] (see also [29] where it appears in a very different context).

We have roughly two different kind of results, depending on the nature of the action of the subgroup of automorphisms involved by the extension: polynomially growing or exponentially growing automorphisms. In Theorem 1 stated below, we extracted two particular cases from the various results of the paper (Theorems 5.1, 5.12, 6.2, 6.23, 7.1) to stress this dichotomy.

Theorem 1. *Let $\mathbb{F}_n$ be the rank n free group, $n \geq 2$.*

(1) Let $\mathcal{U}$ be a finitely generated subgroup of polynomially growing outer automorphisms of $\mathbb{F}_n$. Let μ be a probability measure on $\mathbb{F}_n \rtimes \mathcal{U}$ whose support generates $\mathbb{F}_n \rtimes \mathcal{U}$ as a semi-group. Then there exists a simplicial $\mathcal{U}$-invariant $\mathbb{F}_n$-tree $\mathcal{T}$ such that:

- *Almost every path $\{x_n\}$ of the random walk on $(\mathbb{F}_n \rtimes \mathcal{U}, \mu)$ admits a subsequence which converges to some (random) limit $x_\infty \in \partial\mathcal{T}$.*
- *The distribution of x_∞ is a non-atomic measure λ on $\partial\mathcal{T}$; it is the unique μ-stationary measure on $\partial\mathcal{T}$.*
- *If μ has finite first moment, then the measured space $(\partial\mathcal{T}, \lambda)$ is the Poisson boundary of $(\mathbb{F}_n \rtimes \mathcal{U}, \mu)$.*

A strong version of the conclusions above holds in the case of a direct product $\mathbb{F}_n \times \mathbb{F}_k$: it is useless to pass to a subsequence in the first point and the measure μ in the third point is only required to have finite first logarithmic moment and finite entropy.

(2) Let α be an exponentially growing automorphism of $\mathbb{F}_n$. Let μ be a probability measure on $\mathbb{F}_n \rtimes_\alpha \mathbb{Z}$ whose support generates $\mathbb{F}_n \rtimes_\alpha \mathbb{Z}$ as a semi-group. Then there exists a α-projectively invariant $\mathbb{F}_n$-tree $\mathcal{T}$ such that:

- *Almost every path $\{x_n\}$ of the random walk converges to some limit $x_\infty \in \widehat{\mathcal{T}}$ (where $\widehat{\mathcal{T}} = \mathcal{T} \cup \partial\mathcal{T}$ equipped with the observers topology).*
- *The distribution of x_∞ is a non-atomic measure λ on $\widehat{\mathcal{T}}$; it is the unique μ-stationary measure on $\widehat{\mathcal{T}}$.*
- *If μ has finite first logarithmic moment and finite entropy, then the measured space $(\widehat{\mathcal{T}}, \lambda)$ is the Poisson boundary of $(\mathbb{F}_n \rtimes_\alpha \mathbb{Z}, \mu)$. The measure λ is not concentrated on $\partial\mathcal{T}$.*

We describe the Poisson boundaries with more precision in the full statements given farther in the paper. Note that, in the case of an extension by polynomially growing automorphisms, we need to pass to a finite index subgroup. That is the reason why we require the measure μ to have a finite first moment, and why we obtain the convergence only for a subsequence of a.e. path. See Theorem 1.5 below for more details. We also deal with more general hyperbolic groups than the free group. In the case of an exponentially growing automorphism, the reader will notice that the Poisson boundary is (a quotient of) the whole $\mathbb{R}$-tree, and not only the boundary. At least in the hyperbolic case, this won't seem too surprising for geometric group theorists aware both of the $\mathbb{R}$-tree theory developed for surface and free group automorphisms, and of the existence of the Cannon-Thurston map. Indeed the whole $\mathbb{R}$-tree is homeomorphic to a quotient of $\partial\mathbb{F}_n$ obtained by identifying the points which are the endpoints of a same leaf of a certain "stable lamination" [19]. The quotient we take amounts to further identifying the points of $\partial\mathbb{F}_n$ which are the endpoints of a same leaf of a certain "unstable lamination". That this gives the geometric boundary is known in the case of the suspension of a closed hyperbolic surface by a pseudo-Anosov homeomorphism. In the more general setting we work here we know no reference. It could perhaps be alternatively obtained by combining the Cannon-Thurston map defined in [60] for the relatively hyperbolic setting with the last section of the present paper and a work (yet to be written) of Coulbois-Hilion-Lustig.

Before concluding this introduction we would like to comment on the method and possible generalizations of the results exposed here. First, as was already written, we work really on the $\mathbb{R}$-trees on which the considered groups act, not on the groups themselves. This leads us to extract the properties that we really need for these actions to provide us with a compactification satisfying all the Kaimanovich properties. In particular the map $\mathcal{Q}$ introduced in [55, 56] appears to play a crucial rôle. In fact this map $\mathcal{Q}$ also allows us to get the following

Corollary 1. *Let $\mathbb{F}_n$ be the rank n free group, $n \geq 2$.*

(1) Let $\mathcal{G}$ be an extension of the free group $\mathbb{F}_n$ by a finitely generated subgroup of polynomially growing outer automorphisms. Let μ be a probability measure on $\mathcal{G}$ the support of which generates $\mathcal{G}$ as a semi-group. Then:
   - *There exists a topology on $\mathcal{G} \cup \partial\mathbb{F}_n$ such that almost every path $\{x_n\}$ of the random walk admits a subsequence which converges to some $x_\infty \in \partial\mathbb{F}_n$.*
   - *The distribution of x_∞ is a non-atomic measure λ on $\partial\mathbb{F}_n$; it is the unique μ-stationary measure on $\partial\mathcal{T}$.*
   - *If μ has finite first moment, then the measured space $(\partial\mathbb{F}_n, \lambda)$ is the Poisson boundary of $(\mathcal{G}, \mu)$.*

(2) Let $\mathcal{G}$ be either a cyclic extension of the free group $\mathbb{F}_n$ over an exponentially growing automorphism α of $\mathbb{F}_n$, or a direct product $\mathbb{F}_n \times \mathbb{F}_k$. Let μ be a probability measure on $\mathcal{G}$ whose support generates $\mathcal{G}$ as a semi-group. Then:
   - *There exists a topology on $\mathcal{G} \cup \partial\mathbb{F}_n$ such that almost every path $\{x_n\}$ of the random walk converges to some $x_\infty \in \partial\mathbb{F}_n$.*
   - *The distribution of x_∞ is a non-atomic measure λ on $\partial\mathbb{F}_n$; it is the unique μ-stationary measure on $\partial\mathcal{T}$.*
   - *If μ has finite first logarithmic moment and finite entropy, then the measured space $(\partial\mathbb{F}_n, \lambda)$ is the Poisson boundary of $(\mathcal{G}, \mu)$.*

Beware that $\mathcal{G} \cup \partial\mathbb{F}_n$ is *not* a compactification of $\mathcal{G}$. We invite the reader to compare the above result with [68][Theorems 1 and 2]. A perhaps more intuitive interpretation of the above result in the case of a random walk on a semi-direct product $\mathbb{F}_n \rtimes_\alpha \mathbb{Z}$ is as follows: let $\mathbb{F}_n$ be the rank n free group together with a basis $\mathcal{B}$ and let α be an automorphism of

$\mathbb{F}_n$. Consider a set of transformations $\mathcal{S}$ which consist either of a right-translation by an element in $\mathcal{B}$ or of the substitution of an element g by its image $\alpha(g)$ or $\alpha^{-1}(g)$. Then iterating randomly chosen transformations among the set $\mathcal{S}$ amounts to performing a nearest neighbor random walk on $\mathbb{F}_n \rtimes_\alpha \mathbb{Z}$. The above corollary means that the boundary behavior of this random process is described in terms of the Gromov boundary of $\mathbb{F}_n$.

There is a difficulty, when dealing with direct products or extensions over polynomially growing automorphisms, which does not appear in the exponentially growing case and is occulted in the statements given here. Indeed, whereas in the exponentially growing case we only need to make act the considered group on a $\mathbb{R}$-tree, in the polynomially growing case we have to make act the group on the product of two (simplicial) $\mathbb{R}$-trees. The reason is that, in order to check the Kaimanovich properties (more precisely the (CP) condition), we need that a single element do not fix more than one or two points, and if it fixes two then it acts as a hyperbolic translation along the axis joining the two points. This is not so difficult in the exponentially growing case because there is exactly one fixed point in the interior of the tree, the other ones lying in the Gromov boundary. In the polynomially growing case, some elements fix non-trivial subtrees of the considered $\mathbb{R}$-tree. Collapsing these subtrees is not possible because it might happen that eventually everything gets identified.

The trick is then to make appear, as an intermediate step, a product of trees instead of a single one. Since a semi-direct product structure only depends on the outer-class of the automorphisms, we make the extension considered act in different ways on two copies of the given $\mathbb{R}$-tree. In this way, the fixed points of the action which are an obstruction to some of the Kaimanovich conditions become "small", in some sense, in the ambient space. We mean that there is still a tree to be identified to a single point for some elements but these trees (in $\partial(\widehat{\mathcal{T}} \times \widehat{\mathcal{T}})$) are disjoint for two distinct elements. We come back to a single tree by projection on the first factor.

On the other hand, when considering a group $G \rtimes \mathcal{U}$, the $\mathbb{R}$-trees we work with are only a reminiscence of certain invariant laminations for the action of $\mathcal{U}$ on G. Here the word "invariant lamination" has to be understood in the sense of a geometric lamination or in the more general sense of an algebraic lamination as in [20, 21, 22]. Even in the restricted geometric setting, such invariant laminations might exist even if the manifolds considered are not suspensions. Thus one can expect to be able to find more general classes of groups than those considered here, like for instance in a first step the fundamental groups of compact 3-manifolds which do not fiber over $\mathbb{S}^1$ but nevertheless admit pseudo-Anosov flows (in the decomposition $G \rtimes \mathcal{U}$ we do not need a priori that G and $\mathcal{U}$ be finitely generated, but only that $G \rtimes \mathcal{U}$ is).

1. Kaimanovich descriptions of the Poisson boundary

In this section, we recall the definition given by V.Kaimanovich of the Poisson boundary of a countable group endowed with a probability measure, together with a geometric characterization of this boundary. Most of the material of this section is taken from [48].

We write $\mathbb{N}$ for the set of nonnegative integers. A (measurable) G-space is any measurable space $(X, \mathcal{F})$ measurably acted upon by a countable group G. If μ and λ are measures respectively on G and X, we denote by $\mu * \lambda$ the measure on X which is the the image of the product measure $\mu \otimes \lambda$ by the action $G \times X \to X$. The measure λ is said to be *μ-stationary* if one has:

$$\lambda = \mu * \lambda = \sum_g \mu(g) g\lambda \ . \tag{1}$$

Let G be a countable group, and μ a probability measure on G. The (right) *random walk* on G determined by the measure μ is the Markov chain on G with the transition probabilities $p(x,y) = \mu(x^{-1}y)$ invariant with respect to the left action of the group G on itself. Thus, the position x_n of the random walk at time n is obtained from its position x_0 at time 0 by multiplying by independant μ-distributed right increments h_i:

$$x_n = x_0 h_1 h_2 \cdots h_n. \tag{2}$$

Denote by $G^{\mathbb{N}}$ the space of sample paths $\mathbf{x} = \{x_n\}, n \in \mathbb{N}$ endowed with the σ-algebra $\mathcal{A}$ generated by the cylinders $\{\mathbf{x} \in G^{\mathbb{N}} \mid x_i = g\}$. The group G acts coordinate-wisely on the space $G^{\mathbb{N}}$. An initial distribution θ on G determines the Markov measure $\mathbf{P}_\theta$ on the path space $G^{\mathbb{N}}$ which is the image of the measure $\theta \otimes \bigotimes_{n=1}^{\infty} \mu$ under the map (2). The one-dimensional distribution of $\mathbf{P}_\theta$ at time n, i.e. the distribution of x_n, is $\theta * \mu^{*n}$.

In the sequel, we will be mainly interested in random walks starting at the group identity e which correspond to the initial distribution $\theta = \delta_e$. We denote by $\mathbf{P}$ the associated Markov measure. For any initial distribution θ, one easily checks that the probability measure $\mathbf{P}_\theta$ is equal to $\theta * \mathbf{P}$ and is absolutely continuous with respect to the σ-finite measure $\mathbf{P}_m$, where m is the counting measure on G.

The measure $\mathbf{P}_m$ on the path space $G^{\mathbb{N}}$ is invariant by the time shift $T : \{x_n\} \mapsto \{x_{n+1}\}$. The *Poisson boundary* Γ of the random walk (G, μ) is defined as being the space of ergodic components of the shift T acting on the Lebesgue space $(G^{\mathbb{N}}, \mathcal{A}, \mathbf{P}_m)$ (see [65]).

Let us give some details. Denote by $\sim$ the orbit equivalence relation of the shift T on the path space $G^{\mathbb{N}}$:

$$\mathbf{x} \sim \mathbf{x}' \iff \exists\, n, n' \geq 0,\ T^n \mathbf{x} = T^{n'} \mathbf{x}' \ . \tag{3}$$

Let $\mathcal{A}_T$ be the σ-algebra of all the measurable unions of $\sim$-classes, i.e. the σ-algebra of all T-invariant measurable sets. Denote by $\overline{\mathcal{A}}_T$ the completion of $\mathcal{A}_T$ with respect to the measure $\mathbf{P}_m$. Since $(G^{\mathbb{N}}, \mathcal{A}, \mathbf{P}_m)$ is a Lebesgue space, the Rohlin correspondence assigns to the complete σ-algebra $\overline{\mathcal{A}}_T$ a measurable partition η of $G^{\mathbb{N}}$ called the *Poisson partition*, which is well defined mod 0. An atom of η is an ergodic component of the shift T, that is, up to a set of $\mathbf{P}_m$-measure 0, closed under $\sim$, $\mathcal{A}$-measurable, and minimal with respect to these propoerties. (Note that the σ-algebra $\mathcal{A}_T$ is not *a priori* generated by the atoms of η, see [65, 6]). The Poisson boundary Γ is the quotient space $G^{\mathbb{N}}/\eta$. The coordinate-wise action of G on the path space $G^{\mathbb{N}}$ commutes with the shift T and therefore projects to an action on Γ. Denote by bnd the canonical map $\mathrm{bnd} : G^{\mathbb{N}} \to \Gamma$. The space Γ endowed with the bnd-image of the σ-algebra $\overline{\mathcal{A}}_T$ and the measure $\nu_m = \mathrm{bnd}(\mathbf{P}_m)$ is a Lebesgue space.

For any initial distribution θ, we set $\nu_\theta = \mathrm{bnd}(\mathbf{P}_\theta)$. The Poisson boundary Γ, which depends only on G and μ, carries all the probability measures ν_θ. The measure $\nu = \mathrm{bnd}(\mathbf{P})$ is called the *harmonic measure*. One easily checks that $\nu_\theta = \mathrm{bnd}(\theta * \mathbf{P}) = \theta * \nu$. It implies that the measure ν is μ-stationary, i.e. $\nu = \mu * \nu$ (whereas the other measures ν_θ are not).

As mentioned in the introduction, the space (Γ, ν) enables to retrieve both all the bounded μ-harmonic functions on G and (part of) the asymptotic behaviour of the paths of the random walk. Let us now make this precise.

The Markov operator $P = P_\mu$ of averaging with respect of the transition probability of the random walk (G, μ) is defined by

$$P_\mu f(x) = \sum_y p(x,y) f(y) = \sum_h \mu(h) f(xh) \,. \tag{4}$$

A function $f : G \to \mathbb{R}$ is called μ-harmonic if $Pf = f$. Denote by $H^\infty(G, \mu)$ the Banach space of bounded μ-harmonic functions on G equipped with the sup-norm.

There is a simple way to build bounded μ-harmonic functions on G. Assume that one is given a probability G-space $(X, \mathcal{F}, \lambda)$ such that the measure λ is μ-stationary. To any function $F \in L^\infty(X, \lambda)$ one can assign a function f on G defined, for $g \in G$, by the *Poisson formula* $f(g) = \langle F, g\lambda\rangle = \int_X F(gx) d\lambda(x)$. Since $\lambda = \mu * \lambda$, the function f is μ-harmonic.

In the case of the Poisson boundary (Γ, ν), the Poisson formula is indeed an isometric isomorphism from $L^\infty(\Gamma, \nu)$ to $H^\infty(G, \mu)$. Actually, one can prove (see [47], theorem 6.1) that if f is the harmonic function provided by the Poisson formula applied to a function $F \in L^\infty(\Gamma, \nu)$, then for $\mathbf{P}$-almost every path $\mathbf{x} = \{x_n\}$, one has $F(\mathrm{bnd}\, \mathbf{x}) = \lim f(x_n)$. Conversely, if f is any bounded μ-harmonic functions on G, then the sequence of its values along sample paths $\mathbf{x} = \{x_n\}$ of the random walk is a martingale (with respect to the increasing filtration of the coordinate σ-algebras in $G^{\mathbb{N}}$). Therefore, by the Martingale Convergence Theorem, for $\mathbf{P}$-almost every path $\mathbf{x} = \{x_n\}$, there exists a limit $\hat{F}(\mathbf{x}) = \lim f(x_n)$. This function $\hat{F}$ is T-invariant and $\mathcal{A}_T$-measurable. Since the Poisson boundary Γ is the quotient of the path space determined by $\mathcal{A}_T$, there exists $F \in L^\infty(\Gamma, \nu)$ such that $F(\mathrm{bnd}\, \mathbf{x}) = \lim f(x_n)$.

As far as the behaviour of sample paths at infinity is concerned, we first recall the notion of *μ-boundary*. This notion was first introduced by Furstenberg, see [32, 33]. The following definition is due to Kaimanovich. A μ-boundary is a G-equivariant quotient (B, λ) of the Poisson boundary (Γ, ν), i.e. the quotient of Γ with respect to some G-equivariant measurable partition. The Poisson boundary is itself a μ-boundary, and this space is maximal with respect to this property. Particularly, if π is any T-invariant G-equivariant map from the path space $(G^{\mathbb{N}}, \mathbf{P})$ to some G-space B, then, by definition of the Poisson boundary, such a map factors through Γ: $\pi : G^{\mathbb{N}} \to \Gamma \to B$ and $(B, \pi(\mathbf{P}))$ is a μ-boundary.

Furstenberg's construction of μ-boundaries runs as follows. Assume that B is a separable compact G-space on which G acts continuously. By compactness, there exists a μ-stationary measure λ on B (see [49]). Then the Martingale Convergence Theorem implies that for $\mathbf{P}$-almost every path $\mathbf{x} = \{x_n\}$, the sequence of translations $x_n\lambda$ converges weakly to some measure $\lambda(\mathbf{x})$. Therefore the map $\mathbf{x} \to \lambda(\mathbf{x})$ allows to consider the space of probability measures on B as a μ-boundary. If, in addition, the limit measures $\lambda(\mathbf{x})$ are Dirac measures, then the space B is itself a μ-boundary (see [32, 33]).

For instance, assume that the group G contains the free group $\mathbb{F}_n$ as a normal subgroup. Then the action of G on $\mathbb{F}_n$ by inner automorphisms extends to a continuous action on the boundary $\partial\mathbb{F}_n$ of $\mathbb{F}_n$. Vershik and Malyutin [68] proved that if μ is any nondegenerate measure on G (i.e. the support of μ generates G as a semigroup) then there exists a unique μ-stationary measure λ on the G-space $\partial\mathbb{F}_n$ and $(\partial\mathbb{F}_n, \lambda)$ is a μ-boundary of the pair (G,μ). The aim of this paper is to provide various examples for which we can prove that $(\partial\mathbb{F}_n,\lambda)$ is indeed the Poisson boundary of (G,μ).

The situation we are mainly concerned with is the following. Let $\overline{G}$ be a *compactification* of the group G, that is a topological compact space which contains G as an open dense subset and such that the left-action of G on itself extends to a continuous action on $\overline{G}$. Assume that $\mathbf{P}$-almost every path $\mathbf{x} = \{x_n\}$ converges to a limit $x_\infty = \lim x_n = \pi(\mathbf{x}) \in \overline{G}$. Then the map π is obviously T-invariant and G-equivariant, so that the space $\overline{G}$ equipped with the hitting measure $\pi(\mathbf{P})$ is a μ-boundary. Moreover, in this case, the Poisson formula yields an isometric embedding of $L^\infty(\overline{G},\pi(\mathbf{P}))$ into $H^\infty(G,\mu)$.

A compactification $\overline{G}$ is *μ-maximal* if almost every sample path $\mathbf{x} = \{x_n\}$ of the random walk (G,μ) converges in this compactification, so that $\overline{G}$ is a μ-boundary, and if this μ-boundary is isomorphic to the Poisson boundary (Γ,ν).

In [48], Kaimanovich proves a theorem which provides compactifications $\overline{G}$ of G containing the limit of $\mathbf{P}$-almost every path (theorem 2.4), and another one which is a geometric criterion of maximality of μ-boundaries (theorem 6.5). The combination of these two results yields μ-maximal compactifications of G, see Theorem 1.3 below. This statement will be our central tool.

A compactification $\overline{G}$ is *compatible* if the left-action of G on itself extends to an action on $\overline{G}$ by homeomorphisms. A compactification is *separable* if, when writing $\overline{G} = G\cup\partial G$, then ∂G is separable (i.e contains a countable dense subset). In the remaining of this section, we shall always assume that $\overline{G}$ is a compatible and separable compactification of the group G.

We first state a uniqueness criterion. Its proof is mostly inspired from the one of Theorem 2.4 in [48].

Lemma 1.1. *Let $\overline{G} = G\cup\partial G$ be a compactification of a finitely generated group G satisfying the following* proximality property: *whenever $G \ni g_n^\pm \to \xi^\pm \in \partial G$, one has $g_n\eta \to \xi^+$ for every $\eta \in \partial G\setminus\{\xi^-\}$. Let μ be a probability measure on G such that the subgroup $gr(\mu)$ generated by its support fixes no finite subset of ∂G. Assume that $\mathbf{P}$-almost every path $\mathbf{x} = \{x_n\}$ converges to a (random) limit $x_\infty = \pi(\mathbf{x}) \in \partial G$. Then the hitting measure $\lambda = \pi(\mathbf{P})$, which is the distribution of x_∞, is non-atomic and is the unique μ-stationary probability measure on ∂G.*

Proof. Let ν be any μ-stationary measure on ∂G. The hypothesis on $gr(\mu)$ ensures that ν is non-atomic (see the proof of Theorem 2.4 in [48] for details). Then the proximality property implies that the sequence of (random) measures $x_n\nu$ weakly converges to the Dirac measure δ_{x_∞}. Since ν is μ-stationary and μ^{*n} is the distribution of x_n, we have

$$\nu = \mu^{*n} * \nu = \sum_g \mu^{*n}(g).g\nu = \int x_n\nu d\mathbf{P}(\mathbf{x}) .$$

Passing to the limit on n gives

$$\nu = \int \delta_{x_\infty} d\mathbf{P}(\mathbf{x}) = \lambda .$$

□

Remark 1.2. Note that there is no need of compactness in the proof: Lemma 1.1 remains valid if $\overline{G}$ is not compact. This observation will be used later in the paper.

The compactification $\overline{G}$ satisfies *condition (CP)* if, for any $x \in G$ and for every sequence $g_n \in G$ converging to a point from ∂G in the compactification $\overline{G}$, the sequence $g_n x$ converges to the same limit.

The compactification $\overline{G}$ satisfies *condition (CS)* if the following holds. The boundary ∂G consists of at least 3 points, and there is a G-equivariant Borel map S assigning to pairs of distinct points (b_1, b_2) from ∂G nonempty subsets (*strips*) $S(b_1, b_2) \subset G$ such that for any 3 pairwise distinct points $\overline{b_i} \in \partial G$, $i = 0, 1, 2$, there exist neighbourhoods $\overline{b_0} \in \mathcal{O}_0 \subset \overline{G}$ and $\overline{b_i} \in \mathcal{O}_i \subset \partial G$, $i = 1, 2$ with the property that

$$S(b_1, b_2) \cap \mathcal{O}_0 = \emptyset \qquad \text{for all} \quad b_i \in \mathcal{O}_i,\ i = 1, 2\,.$$

This condition (CS) means that points from ∂G are separated by the strips $S(b_1, b_2)$.

A *gauge* on a countable group G is any increasing sequence $\mathcal{J} = (\mathcal{J}_k)_{k \geq 1}$ of sets exhausting G. The corresponding *gauge function* is defined by $|g| = |g|_{\mathcal{J}} = \min\{k\,,\, g \in \mathcal{J}_k\}$. The gauge $\mathcal{J}$ is *finite* if all gauge sets are finite. An important class of gauges consists of *word gauges*, i.e. gauges $(\mathcal{J}_k)$ such that $\mathcal{J}_1$ is a set generating G as a semigroup, and $\mathcal{J}_k = (\mathcal{J}_1)^k$ is the set of word of length $\leq k$ in the alphabet $\mathcal{J}_1$. It is finite if and only if $\mathcal{J}_1$ is finite. A set $S \subset G$ *grows polynomially* with respect to some gauge $\mathcal{J}$ if

$$\text{there exist } A, B, d \text{ such that } \mathit{card}\,[S \cap \mathcal{J}_k] \leq A + Bk^d\,.$$

Theorem 1.3 ([48], Theorems 2.4 and 6.5)**.** *Let G be a finitely generated group. Let $\overline{G} = G \cup \partial G$ be a compatible and separable compactification of G satisfying conditions (CP) and (CS). Let μ be a probability measure on G such that the subgroup $gr(\mu)$ generated by its support fixes no finite subset of ∂G.*

Then $\mathbf{P}$-almost every path $\mathbf{x} = \{x_n\}$ converges to a (random) limit $x_\infty = \pi(\mathbf{x}) \in \partial G$. The hitting measure $\lambda = \pi(\mathbf{P})$ is non-atomic, the measure space $(\partial G, \lambda)$ is a μ-boundary and λ is the unique μ-stationary probability measure on ∂G.

Moreover, if $\mathcal{J}$ is a finite word gauge on G such that the measure μ has a finite first logarithmic moment $\sum \log|g|\mu(g)$, a finite entropy $H(\mu) = -\sum \mu(g)\log\mu(g)$, and each strip $S(b_1, b_2)$ grows polynomially, then the space $(\partial G, \lambda)$ is isomorphic to the Poisson boundary (Γ, ν) of (G, μ) and is therefore μ-maximal.

Note that if a measure μ has a finite first moment $\sum |g|\mu(g)$, then the entropy $H(\mu)$ is finite, see e.g. [47], Lemma 12.2. Besides, the finitess of the first (logarithmic) moment does not depend on the choice of the finite word gauge.

We end this section with a discussion on the stability of the Poisson boundary when moving to a subroup which is recurrent or normal with finite index. Let G be a finitely generated group, μ a probability measure on G and G^0 a subgroup of G which is a recurrent set for the random walk (G, μ). Define a probability measure μ^0 on G^0 as the distribution of the point where the random walk issued from the identity of G returns for the first time to G^0. We call μ^0 the *first return measure.* Furstenberg observed ([32], Lemma 4.2) that the Poisson boundaries of (G, μ) and (G^0, μ^0) are isomorphic.

For instance, if G^0 is a normal subgroup in G such that the random walk (G, μ) projects on the factor group G/G^0 as a recurrent random walk, then the identity in G/G^0 is a recurrent state, therefore G^0 is a recurrent set in G. The case of a normal subgroup of finite index is of special interest:

Lemma 1.4 ([32], Lemma 4.2 and [45], Lemma 2.3). *Let G be a finitely generated group, μ a probability measure on G and G^0 a normal subgroup of finite index in G. Then G^0 is a recurrent set for the random walk (G, μ) and the Poisson boundaries of (G, μ) and (G^0, μ^0) are isomorphic where μ^0 is the first return measure. Moreover, if μ has a finite first moment (with respect to some finite word gauge) then so has μ^0.*

Observe that the conclusion of this lemma remains valid if the finite index subgroup G^0 is not normal, since it contains a subroup G^1 of finite index which is normal in G.

Indeed, what is meant in this lemma is that there is an isomorphism between the spaces of harmonic functions $H^\infty(G, \mu)$ and $H^\infty(G^0, \mu^0)$. But the Poisson boundaries (Γ, ν) and (Γ^0, ν^0) of (G, μ) and (G^0, μ^0) are distincts objects in nature since Γ^0 is not *a priori* a G-space. This is made precise in the following statement, which is a combination of Theorem 1.3 and (the proof of) Lemma 1.4:

Theorem 1.5. *Let G be a finitely generated group and G^0 a finite index subgroup of G. Let $\overline{G^0} = G^0 \cup B$ be a compatible and separable compactification of G^0 satisfying conditions (CP) and (CS) such that the action of G^0 on B extends to a continuous action of G on B. Let μ be a probability measure on G such that the subgroup $gr(\mu)$ generated by its support fixes no finite subset of B. Denote by μ^0 the first return measure. Let τ_k be the (random) sequence of the times at which the path $\mathbf{x} = \{x_n\}$ visits G^0. Then :*

(1) $\mathbf{P}$-almost every sub-path x_{τ_k} converges to a (random) limit $x_\infty = \pi(\mathbf{x}) \in B$.

(2) the hitting measure $\lambda = \pi(\mathbf{P})$ is non-atomic, the measure space (B, λ) is a μ-boundary and λ is the unique μ-stationary probability measure on B.

(3) if $\mathcal{J}$ is a finite word gauge on G such that the measure μ has a finite first moment and each strip grows polynomially, then the space (B, λ) is isomorphic to the Poisson boundary (Γ, ν) of (G, μ).

Proof. Since G^0 has finite index in G, the subgroup $gr(\mu^0)$ of G^0 generated by the support of μ^0 fixes no finite subset of B. Therefore Theorem 1.3 works for the random walk (G^0, μ^0), which gives Item (1), as well as Items (2) and (3) for μ^0.

Let us prove that λ is the unique μ-stationary probability measure on B. Let ν be any μ-stationary probability measure on B (the compactness of B implies the existence of such a measure). Consider the Poisson formula $\Pi_\nu : L^\infty(B, \nu) \to H^\infty(G, \mu)$. According to the proof of Lemma 4.2 in [32], the restriction $\Phi : f \mapsto f_{|G^0}$ maps μ-harmonic functions to μ^0-harmonic functions. Therefore the composition $\Phi \circ \Pi_\nu$ is the Poisson formula $L^\infty(B, \nu) \to H^\infty(G^0, \mu^0)$ so the measure ν is μ^0-stationary, and $\nu = \lambda$.

As far as Item (3) is concerned, we prove that the Poisson formula $\Pi_\lambda : L^\infty(B, \lambda) \to H^\infty(G, \mu)$ is an isomorphism. The map $\Phi : H^\infty(G, \mu) \to H^\infty(G^0, \mu^0)$ is an isomorphism. Since μ has a first moment, so has μ^0. According to Item (3) of Theorem 1.3, (B, λ) is the Poisson boundary of (G^0, μ^0), therefore the composition $\Phi \circ \Pi_\lambda$ is an isomorphism, and so is Π_λ. $\square$

Observe that the way the group G acts on B does not play any role, provided this action extends the action of G^0. Indeed, the boundary behaviour of the random walk on (G, μ) is governed by (G^0, μ^0).

Assume now the following: G^0 is the free group $\mathbb{F}_n$, the factor group $G/\mathbb{F}_n$ is isomorphic to $\mathbb{Z}$ and the image random walk on $G/\mathbb{F}_n$ is recurrent. Then the above construction gives rise to a probability measure μ^0 on $\mathbb{F}_n$ such that the Poisson boundaries of (G, μ) and $(\mathbb{F}_n, \mu^0)$ are isomorphic. The problem is that, in this case, we do not know *a priori* what the Poisson boundary of $(\mathbb{F}_n, \mu^0)$ should be, because the measure μ^0 is potentially too spread out to have a finite entropy, even if the image random walk is the symmetric nearest neighbour random walk on $\mathbb{Z}$. On the other hand, according to Corollary 1, the Poisson boundary of (G, μ) - and that of $(\mathbb{F}_n, \mu^0)$ - is $\partial\mathbb{F}_n$.

Besides, if G is, for instance, an extension of the free group $\mathbb{F}_4$ by a $\mathbb{Z}^2$-subgroup of polynomially growing automorphisms in $\mathrm{Aut}(\mathbb{F}_4)$ (see Subsection 8.2 for such examples) and if μ is any non-degenerate probability measure with finite first moment on G which leads to a recurrent random walk on $G/\mathbb{F}_4 \sim \mathbb{Z}^2$ then the measure μ^0 on $\mathbb{F}_4$ is potentially *very* spread out, and Corollary 1 ensures that the Poisson boundary of $(\mathbb{F}_4, \mu^0)$ is $\partial\mathbb{F}_4$.

2. Generalities about groups, automorphisms and semi-direct products

Let G be a discrete group with generating set S, that we denote by $G = \langle S \rangle$. The Cayley graph of G with respect to S is denoted by $\Gamma_S(G)$. It is equipped with the standard metric which makes each edge isometric to $(0, 1)$. We denote by $|\gamma|_S$ the word-length of an element γ with respect to S, i.e. the minimal number of elements in $S \cup S^{-1}$ necessary to write γ. If it is clear, or not important, which generating set is used, we will simply write $|\gamma|$ for the word-length of $\gamma \in G$ (in particular, when dealing with free groups, $|\gamma|$ will denote the word-length with some fixed basis).

We denote by $\mathrm{Aut}(G)$ the group of automorphisms of G, by $\mathrm{Inn}(G)$ the group of inner automorphisms (i.e. automorphisms of the form $i_g(x) = gxg^{-1}$) and by $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ the group of outer automorphisms. While $\mathrm{Aut}(G)$ acts on elements of G, $\mathrm{Out}(G)$ acts on conjugacy-classes of elements. If $\mathcal{U} < \mathrm{Aut}(G)$ is a subgroup, we denote by $[\mathcal{U}] < \mathrm{Out}(G)$ its image under the canonical projection.

Definition 2.1. An automorphism α of a G has *polynomial growth* if there is a polynomial function P such that, for any γ in G, for any $m \in \mathbb{N}$, $|\alpha^m(\gamma)| \leq P(m)|\gamma|$.

We say that α has *exponential growth* if the lengths of the iterates $\alpha^m(\gamma)$ of at least one element γ grow at least exponentially with $m \to +\infty$.

The above notions only depend on the outer-class of automorphisms considered. Passing from α to α^{-1} neither changes the nature of the growth.

Let $\theta \colon \mathcal{U} \to \mathrm{Aut}(G)$ be a monomorphism. We denote by $G_\theta := G \rtimes_\theta \mathcal{U}$ the semi-direct product of G with $\mathcal{U}$ over θ. The semi-direct product only depends on the outer-class of $\theta(\mathcal{U})$: G_θ is isomorphic to $G_{\theta'}$ whenever $[\theta(\mathcal{U})] = [\theta'(\mathcal{U})]$ in $\mathrm{Out}(G)$. For this reason we will also write a semi-direct product $G_\theta := G \rtimes_\theta \mathcal{U}$ for $\theta \colon \mathcal{U} \to \mathrm{Out}(G)$.

We will denote by $\mathbb{F}_n = \langle x_1, \cdots, x_n \rangle$ the rank n free group. A *hyperbolic group* is a finitely generated group $G = \langle S \rangle$ such that there exists $\delta \geq 0$ for which the geodesic triangles of $\Gamma_S(G)$ are δ-thin:

For any triple of geodesics $[x, y], [y, z], [x, z]$ in $\Gamma_S(G)$, $[x, z] \subset \mathcal{N}_\delta([x, y] \cup [y, z])$, where $\mathcal{N}_\delta(X)$ denotes the set of all points at distance smaller than δ from some point in X.

We briefly recall the definition of the Gromov boundary ∂G of a hyperbolic group G [41], which allows one to get a geodesic compactification $G \cup \partial G$ of G. For more details, see [50]. The Gromov boundary ∂G of the hyperbolic group G consists of the set of equivalence classes of geodesic rays emanating from the base-point, two rays being equivalent if their Hausdorff distance is finite. A *ray emanating from O* is the interior of (the image of) an isometric embedding $\rho \colon [0, +\infty) \to \Gamma_S(G)$ with $\rho(0) = O$. Of course the definition of a Gromov boundary naturally extends to any Gromov hyperbolic metric space (a geodesic metric space whose geodesic triangles are δ-thin). In the particular case of a tree T, which will be frequent in this paper, two geodesic rays in T define a same point in ∂T if they eventually coincide. Beware that the Gromov boundary ∂X of a geodesic metric space X *does not* provide us with a compactification of X if X is not proper (where "proper" here means that the closed balls are compact). For instance, if T is a locally infinite tree, then $T \cup \partial T$ is not a compactification of T.

3. About $\mathbb{R}$-trees and group actions on $\mathbb{R}$-trees

The aim of this section is to gather all the vocabulary as well as all the notions and results that we will need about $\mathbb{R}$-trees and group actions on $\mathbb{R}$-trees further in the paper. The reader may choose to skip this chapter, only coming back each time subsequent sections refers to the material developed below. We counsel however to have a quick look at the first part, which motivates the introduction of these $\mathbb{R}$-trees.

3.1. **How $\mathbb{R}$-trees come into play.** We take a pedestrian way to show to the reader how $\mathbb{R}$-trees naturally come into play when searching for Poisson boundaries of groups extensions. We consider here the group $\mathbb{F}_n \times \mathbb{Z}$. Let $\mathbb{F}_n = \langle x_1, \cdots, x_n \rangle$ and $\mathbb{Z} = \langle t \rangle$. We assume it is equipped with a probability measure μ and we would like to find a compactification of $\mathbb{F}_n \times \mathbb{Z}$ suitable for applying Kaimanovich tools.

The most natural space on which $\mathbb{F}_n \times \mathbb{Z}$ acts is of course $T \times \mathbb{Z}$, where T is the Cayley-graph of $\mathbb{F}_n$ with respect to some basis. This is a simplicial, locally finite tree. For compactifying $T \times \mathbb{Z}$, if v is a vertex of T we need that $t^{k_i}v$ converges to some point O_v for any vertex of T. For the (CP) property to be satisfied, we need that the limit-point O_v be the same for all v. This point would then be fixed by all elements of $\mathbb{F}_n \times \mathbb{Z}$, so that Theorem 1.3 would not apply. The cause of the problem here holds in the fact that the $\mathbb{Z}$-action on $\mathbb{F}_n$ has many fixed points ...

We are now going to take advantage from the fact that a direct product is a particular case of a semi-direct product: $\mathbb{F}_n \times \mathbb{Z}$ is isomorphic to $\mathbb{F}_n \rtimes_\alpha \mathbb{Z}$ with $\alpha \in \mathrm{Inn}(G)$. We define $\alpha \in \mathrm{Inn}(\mathbb{F}_n)$ by $\alpha(x_i) = x_1 x_i x_1^{-1}$ for $i \in \{1, \cdots, n\}$. The Cayley-graph of $\mathbb{F}_n \rtimes_\alpha \mathbb{Z}$ is now the 1-skeleton of $\bigsqcup_{t \in \mathbb{Z}} T \times [t, t+1]/((x, t+1) \in T \times [t, t+1] \sim (f(x), t+1) \in T \times [t+1, t+2])$, where $f \colon T \to T$ is a PL-map realizing α, i.e. the image under f of an edge of T with label x_i is a PL-path in T with associated edge-path $\alpha(x_i)$. We observe that, contrary to what happened before, there are now two kinds of asymptotic behavior for the $\alpha^k(w)$'s: for any $w \neq x_1$, $\lim_{k \to \pm\infty} \dfrac{|\alpha^k(w)|}{|k|} = 2$ whereas $\lim_{k \to \pm\infty} \dfrac{\alpha^k(x_1)}{|k|} = 0$. In other words, for $w \neq x_1$, the length of the $\alpha^k(w)$ tend linearly toward infinity with k whereas the lengths of the $\alpha^k(x_1) = x_1$ remain constant equal to 1.

Having still in the mind the compactification of the Cayley graph, since some of the orbits separate linearly we scale the metric on $T \times \{i\}$ by the factor $\frac{1}{|i|+1}$. Passing to the limit in the sequence of metric spaces

$$(T \times \{0\}, |.|) \to (T \times \{1\}, \frac{|.|}{2}) \to \cdots \to (T \times \{i\}, \frac{|.|}{i+1}) \to \cdots$$

(see [10, 40, 34, 62] for instance) we get a so-called "α-invariant simplicial $\mathbb{F}_n$-tree" $\mathcal{T}$. This is a simplicial tree, equipped with an isometric action of $\mathbb{F}_n$, and which admits moreover an isometry H which "commutes" with α:

$$\text{For any } P \in \mathcal{T},\ H(wP) = \alpha(w)H(P).$$

There are non trivial vertex $\mathbb{F}_n$-stabilizers: these are the conjugates $w\langle x_1 \rangle w^{-1}$. Thus the tree $\mathcal{T}$ is locally infinite. The edge $\mathbb{F}_n$-stabilizers are trivial. Although this last property will be found in all the exploited cases, such a tree is only a very particular case of the $\mathbb{R}$-trees below because of its simplicial nature.

3.2. **$\mathbb{R}$-trees and the observers topology.** A $\mathbb{R}$-tree $(\mathcal{T}, d)$ is a geodesic metric space which is 0-hyperbolic for the associated distance-function d. We often simply write $\mathcal{T}$, the distance-function being implicit. The metric completion of an $\mathbb{R}$-tree still is an $\mathbb{R}$-tree. We denote by $\overline{\mathcal{T}}$ the union of the completion of $\mathcal{T}$ with its Gromov boundary $\partial\mathcal{T}$.

Definition 3.1. [19] Let $\mathcal{T}$ be a complete $\mathbb{R}$-tree.

Let P, Q be any two distinct points in $\overline{\mathcal{T}}$. A *direction at* P, denoted by D_P, is a connected component of $\overline{\mathcal{T}} \setminus \{P\}$. The *direction of* Q *at* P, denoted by $D_P(Q)$, is the connected component of $\overline{\mathcal{T}} \setminus \{P\}$ which contains Q.

A *branch-point* is any point in $\mathcal{T}$ at which there are at least three different directions.

An *extremal point* is a point P at which there is only one direction, i.e. $\overline{\mathcal{T}} \setminus \{P\}$ is connected. The *interior tree* of $\mathcal{T}$ is the tree deprived of its extremal points.

An *arc* is a subset isometric to an interval of the real line. We denote by $[P, Q]$ the geodesic arc from P to Q (P or Q, or both, may belong to $\partial\mathcal{T}$, in which case $[P, Q]$ denotes the unique infinite, or bi-infinite, geodesic between P and Q).

Among the $\mathbb{R}$-trees we distinguish the *simplicial* $\mathbb{R}$*-trees*, defined as the $\mathbb{R}$-trees in which every point which is not a branch-point admits a neighborhood homeomorphic to an open interval. In these simplicial trees, it is thus possible to speak of *vertices* (the branch-points) and of *edges* (the connected components of the complement of the vertices). The metric of the $\mathbb{R}$-tree defines a length on each edge. If there is a uniform lower-bound on the length of the edges, the vertices form a discrete subset of the simplicial $\mathbb{R}$-tree for the metric topology.

Definition 3.2. Let $\mathcal{T}$ be a $\mathbb{R}$-tree. The *observers'topology* on $\overline{\mathcal{T}}$ is the topology which admits as an open neighborhoods basis the set of all directions D_P, $P \in \overline{\mathcal{T}}$, and their finite intersections. We denote by $\widehat{\mathcal{T}}$ the topological space obtained by equipping $\overline{\mathcal{T}}$ with the observers'topology.

The observers topology is weaker than the metric topology [19]. Any sequence of points (Q_n) *turning around a point* P *of* $\mathcal{T}$, meaning that every direction at P contains only finitely many of the Q_n's, converges to P in $\widehat{\mathcal{T}}$. Such a phenomenon is of course only possible in a non locally finite tree. In a locally finite, simplicial tree, the metric and observers topology agree. We summarize below what makes this topology important for us.

Proposition 3.3. [19] *Let* $\mathcal{T}$ *be a separable* $\mathbb{R}$*-tree. Then:*

1. *(1)* $\widehat{\mathcal{T}}$ *is compact.*
2. *(2) The metric topology and the observers'topology agree on the Gromov boundary* $\partial\mathcal{T}$*. In particular* $\partial\mathcal{T} \subset \widehat{\mathcal{T}}$ *is separable.*
3. *(3) If* (P_n) *is a sequence of points in* $\overline{\mathcal{T}}$ *and* Q *is a fixed point in* $\overline{\mathcal{T}}$*, there is a unique point* $P := \liminf_{n\to\infty} {}_Q P_n$ *defined by*

$$[Q, P] = \overline{\bigcup_{m=0}^{\infty} \bigcap_{n \geq m} [Q, P_n]}$$

4. *(4) If* (P_n) *is a sequence of point in* $\widehat{\mathcal{T}}$ *which converges to some point* P *in* $\widehat{\mathcal{T}}$ *then for any* $Q \in \overline{\mathcal{T}}$*,* $P = \liminf_{n\to\infty} {}_Q P_n$ *holds.*

3.3. **Affine actions on $\mathbb{R}$-trees.**

Definition 3.4. Let $(\mathcal{T}, d)$ be a $\mathbb{R}$-tree. A *homothety* $H\colon \mathcal{T} \to \mathcal{T}$ *of dilation factor* $\lambda \in \mathbb{R}^{+*}$ is a homeomorphism of $\mathcal{T}$ such that for any $x, y \in \mathcal{T}$:

$$d(H(x), H(y)) = \lambda d(x, y).$$

The homothety is *strict* if $\lambda \neq 1$.

By the fixed-point theorem, a strict homothety on a complete $\mathbb{R}$-tree admits exactly one fixed point.

Let $\mathcal{T}$ be a $\mathbb{R}$-tree and let H be a homothety of $\mathcal{T}$. Then H induces a permutation on the set of directions of $\mathcal{T}$.

Definition 3.5. [57] Let $\mathcal{T}$ be a $\mathbb{R}$-tree and let H be a homothety on $\mathcal{T}$. An *eigenray of* H *at a point* P is a geodesic ray R starting at P such that $H(R) = R$.

In particular, if R is an eigenray at P, then $H(P) = P$. In [57] it was proven that if a strict homothety on the completion of a $\mathbb{R}$-tree $\mathcal{T}$ has its fixed point outside $\mathcal{T}$, then it fixes a unique eigenray starting at this fixed point. In [34] (among others) it was noticed that if such a homothety leaves invariant a direction D_P (P is thus a fixed point), then it admits a unique eigenray in D_P.

Definition 3.6. Let G be a discrete group which acts by homeomorphisms on a $\mathbb{R}$-tree $(\mathcal{T}, d)$.

The action of G on $\mathcal{T}$ is *irreducible* if there is no finite invariant set in $\overline{\mathcal{T}}$.

The action of G on $\mathcal{T}$ is *minimal* if there is no proper invariant subtree (where a *proper* subset of a set is a subset distinct from the set itself).

The action of G on $\mathcal{T}$ is an *action by homotheties*, or an *affine action*, if there is a morphism $\lambda\colon G \to \mathbb{R}^{+*}$ with $d(hx, hy) = \lambda(h)d(x, y)$ for any $h \in G$ and $x, y \in \mathcal{T}$.

A *G-tree* is a $\mathbb{R}$-tree equipped with an isometric action of G.

We refer to [57] for a specific study of these affine actions. By the very definition, when given an affine action of a discrete group G on a $\mathbb{R}$-tree $\mathcal{T}$, each element is identified to a homothety of $\mathcal{T}$. Thus an affine action of a group G on a $\mathbb{R}$-tree $\mathcal{T}$ induces a morphism from G to the permutations on the set of directions of $\mathcal{T}$.

A common way in which affine actions arise is as follows:

Definition 3.7. Let $\mathcal{T}$ be a G-tree and let $\theta\colon \mathcal{U} \to \mathrm{Aut}(G)$ be a monomorphism.

The G-tree $\mathcal{T}$ is a *$\theta(\mathcal{U})$-projectively invariant G-tree* if $\theta(\mathcal{U})$ acts by homotheties on $\mathcal{T}$ such that for any $g \in G$, for any $u \in \mathcal{U}$ and for any point $P \in \mathcal{T}$:

$$H_{\theta(u)}(g.P) = \theta(u)(g) H_{\theta(u)}(P)$$

If $\theta(\mathcal{U})$ acts by isometries then $\mathcal{T}$ is a *$\theta(\mathcal{U})$-invariant G-tree.*

In the expression "$\theta(\mathcal{U})$-projectively invariant $\mathbb{R}$-tree" we will often substitute the group $\theta(\mathcal{U})$ by its generators. In particular, instead of "$\langle\alpha\rangle$-projectively invariant $\mathbb{R}$-tree", we will rather write "α-projectively invariant $\mathbb{R}$-tree".

Remark 3.8. If $\mathcal{T}$ is a $\theta(\mathcal{U})$-projectively invariant G-tree, then the semi-direct product $G_\theta := G \rtimes_\theta \mathcal{U}$ acts by homotheties on $\mathcal{T}$. Conversely, if one has an affine action of G_θ on a $\mathbb{R}$-tree $\mathcal{T}$ such that the induced action of $G \lhd G_\theta$ is an isometric action, then $\mathcal{T}$ is a $\theta(\mathcal{U})$-projectively invariant G-tree.

When one has a $\theta(\mathcal{U})$-projectively invariant G-tree $\mathcal{T}$, then $\mathcal{T}$ is in fact projectively invariant for any subgroup $\theta'(\mathcal{U}) < \mathrm{Aut}(G)$ with $[\theta(\mathcal{U})] = [\theta'(\mathcal{U})]$. Indeed, let $\theta'(u) \in \mathrm{Aut}(G)$ satisfy $\theta'(u)(.) = g_0\theta(u)(.)g_0^{-1}$ for some $g_0 \in G$. Then $H_{\theta'(u)} = g_0 H_{\theta(u)}$ satisfies $H_{\theta'(u)}(gP) = \theta'(u)(g)H_{\theta'(u)}(P)$. Indeed

$$H_{\theta'(u)}(gP) = g_0 H_{\theta(u)}(g.P) = g_0\theta(u)(g)H_{\theta(u)}(P),$$
$$g_0\theta(u)(g)H_{\theta(u)}(P) = g_0\theta(u)(g)g_0^{-1}g_0H_{\theta(u)}(P) = \theta'(u)(g)g_0H_{\theta(u)}(P)$$

and $g_0 H_{\theta(u)}(P) = H_{\theta'(u)}(P)$ eventually gives the announced equality. This justifies to state the

Definition 3.9. Let $\mathcal{T}$ be a G-tree and let $\theta\colon \mathcal{U} \to \mathrm{Out}(G)$ be a monomorphism.

The G-tree $\mathcal{T}$ is a *$\theta(\mathcal{U})$-projectively invariant G-tree* if there is $\tilde{\theta}\colon \mathcal{U} \to \mathrm{Aut}(G)$ with $[\tilde{\theta}(\mathcal{U})] = \theta(\mathcal{U})$ such that $\mathcal{T}$ is a $\tilde{\theta}(\mathcal{U})$-projectively invariant G-tree.

3.4. **The LL-map $\mathcal{Q}$.** The LL-map $\mathcal{Q}$ which appears below was introduced in [55, 56] in the setting of free group automorphisms (the two L's of the denomination "LL-map" stand for the name of the authors Levitt-Lustig - see also [19] for more details about this map).

Lemma 3.10. *Let G be a discrete group acting by homotheties on a $\mathbb{R}$-tree $\mathcal{T}$. Let $\overline{G} = G \cup \partial G$ be a compatible compactification of G. Assume that the* LL-map

$$\mathcal{Q}\colon \left\{ \begin{array}{ccc} \partial G & \to & \mathcal{T} \\ X & \mapsto & \liminf_{n\to\infty} {}_P g_n P \end{array} \right.$$

is well-defined, that is independent from the point P and the sequence (g_n) converging to X in $\overline{G}$.

Then for any infinite, not eventually constant sequence (g_n) of elements in G such that (g_nP) tends to some point Q in $\widehat{\mathcal{T}}$, for any $R \in \widehat{\mathcal{T}}$, (g_nR) tends to Q in $\widehat{\mathcal{T}}$.

Proof. Let R be any point in $\widehat{\mathcal{T}}$ and consider the sequence (g_nR). By compacity of $\widehat{\mathcal{T}}$ (see Proposition 3.3), it has at least one accumulation point. Consider any convergent subsequence $(g_{n_k}R)$ and denote by Q' its limit. By passing if necessary to a further subsequence we can suppose that (g_{n_k}) tends to some point $X \in \partial G$ in $\overline{G}$. By the assumption that (g_nP) tends toward Q, the very definition of the map $\mathcal{Q}$ implies $\mathcal{Q}(X) = Q$. By Proposition 3.3, $Q' = \liminf_{n\to\infty} {}_R g_{n_k} R$ and since (g_{n_k}) converges to X, we get $Q' = \mathcal{Q}(X) = Q$. We so proved that Q is the only accumulation-point of (g_nR) in $\widehat{\mathcal{T}}$. Since $\widehat{\mathcal{T}}$ is compact, this implies that (g_nR) tends toward Q in $\widehat{\mathcal{T}}$. The lemma follows. □

In the case where G is a hyperbolic group and $\mathcal{T}$ is a simplicial G-tree with quasiconvex vertex stabilizers and trivial edge-stabilizers, the existence of the LL-map $\mathcal{Q}$ is easy to prove. Observe also that the very existence of the map $\mathcal{Q}$ implies the triviality of the arc stabilizers (however, in the cases that we will consider, this triviality is proved in a direct way rather than by appealing to the more complex notion that represents the map $\mathcal{Q}$). Similarly, for the conclusion of Lemma 3.10 to be true, we need the triviality of the arc-stabilizers.

Remark 3.11. Consider a G-tree $\mathcal{T}$ with trivial arc-stabilizers. The stabilizers H_i of the branch-points are *malnormal*: for any $g \in G \setminus H_i$, $g^{-1}H_ig \cap H_i = \{1\}$. Moreover, any family $\mathcal{H}$ of branch-points stabilizers which belong to distinct G-orbits form a *malnormal family* of subgroups: for any $g \in G$, for any $H_i \neq H_j$ in $\mathcal{H}$, $g^{-1}H_ig \cap H_j = \{1\}$.

4. Some classes of groups with affine actions on $\mathbb{R}$-trees

We gather here various results which provide us with non-trivial classes of groups with affine actions on $\mathbb{R}$-trees. Our results about the Poisson boundary of these groups, developed in the next sections, will rely upon the theorems recalled here. Once again the reader can skip for the moment the exposition of these results and only go back here each time the theorems are referred to.

4.1. Projectively invariant $\mathbb{R}$-trees for single automorphisms.

Theorem 4.1. *Let G be the fundamental group of a compact hyperbolic surface S and let $\alpha \in \mathrm{Aut}(G)$ in an outer-class induced by a homeomorphism of S. Assume that α has exponential growth. Then there exists a separable, minimal G-tree $\mathcal{T}_\alpha$ which is α-projectively invariant and satisfies the following properties:*

1. *The action of $G_\alpha = G \rtimes_\alpha \mathbb{Z}$ is irreducible (see Definition 3.6).*
2. *The G-action has dense orbits in $\overline{\mathcal{T}}_\alpha$.*
3. *There are a finite number of G-orbits of branch-points. The stabilizers of branch-points are quasiconvex. For any point P in $\mathcal{T}$, the number of $\mathrm{Stab}_G(P)$-orbits of directions at P is finite.*
4. *The LL-map $\mathcal{Q}\colon \partial G \to \widehat{\mathcal{T}}_\alpha$ is well-defined (in particular the arc-stabilizers are trivial for the induced G-action), continuous, G_α-equivariant and such that the pre-images of any two distinct points are disjoint compact subsets of ∂G. The pre-image of a point in $\partial \mathcal{T}$ is a single point of ∂G.*

We refer the reader to [28] for the basis about isotopy-classes of surface homeomorphisms. Since we know no precise reference for the above theoem, we briefly sketch a proof.

Proof. Let us first briefly describe how one gets a $\mathbb{R}$-tree $\mathcal{T}$ as in Theorem 4.1. We consider a homeomorphism h inducing $[\alpha] \in \mathrm{Out}(G)$ as given by the Nielsen-Thurston classification: there is a decomposition of S in subsurfaces $S_1, \cdots, S_r$ such that $h^{k_i}(S_i) = S_i$ and $h^{k_i}_{|S_i}$ is either pseudo-Anosov or has linear growth. The S_i's for which $h^{k_i}_{|S_i}$ has linear growth are maximal with respect to the inclusion. Each boundary curve of each S_i is fixed by h^{k_i}. Since α has exponential growth, there is at least one subsurface S_i such that $h^{k_i}_{|S_i}$ is pseudo-Anosov. Let S_0 be the subsurface for which $h^{k_0}_{|S_0}$ is pseudo-Anosov with greatest dilatation factor, denoted by λ_0. We now consider the universal covering $\tilde{S}$ of S (this is a hyperbolic Poincaré disc $\mathbb{D}^2$ in the closed case, the complement of horoballs in $\mathbb{D}^2$ in the free group case) and a lift $\tilde{h}$ of h. Each connected component of $\tilde{S}_i$, $i > 0$, each reduction curve and each boundary curve is collapsed to a point. Since the corresponding subgroups are quasi convex and malnormal, the resulting metric space $\hat{S}$ is still a Gromov hyperbolic space (of course not proper). The connected components of $\tilde{S}_0$ are equipped with a pair of invariant, transverse, transversely measured singular foliations $(\mathcal{F}_u, \mu_s)$ and $(\mathcal{F}_s, \mu_u)$: $\tilde{h}((\mathcal{F}_u, \mu_s)) = (\mathcal{F}_u, \frac{1}{\lambda_0}\mu_s)$ and $\tilde{h}((\mathcal{F}_s, \mu_u)) = (\mathcal{F}_s, \lambda_0\mu_u)$. We equip $\hat{S}$ with the length-metric associated to the transverse measures: $|\gamma|_\mathcal{F} = \int_\gamma |d\mu_u| + |d\mu_s|$. The resulting metric space $\hat{S}_\mathcal{F}$ is still Gromov hyperbolic. We now consider the limit-metric space

$$(\hat{S}_\mathcal{F} \times \{0\}, |.|_\mathcal{F}) \to (\hat{S}_\mathcal{F} \times \{1\}, \frac{|.|_\mathcal{F}}{\lambda_0}) \to \cdots \to (\hat{S}_\mathcal{F} \times \{i\}, \frac{|.|_\mathcal{F}}{\lambda_0^i}) \to \cdots$$

obtained by rescaling the metric $|.|_\mathcal{F}$ by λ_0^i for $i \to +\infty$. From [34], the limit-space is a α-projectively invariant G-tree $\mathcal{T}$ such that the G-action has trivial arc-stabilizers

([34] deals with free group automorphisms but, as noticed there, the triviality of the arc-stabilizers is proved in the same way when considering a hyperbolic group). Possibly after restricting to a minimal subtree, the G-action has dense orbits, see for instance [62]: this is true as soon as the action of α is by strict homotheties and the $\mathbb{R}$-tree is minimal.

Claim: Let $\mathcal{L}_s$ be the geodesic lamination associated to $\mathcal{F}_s$. Each non-boundary leaf of $\mathcal{L}_s$ is mapped to a point in $\mathcal{T}$ which is not a branch-point. Any two points in $\hat{S}_{\mathcal{F}}$ on distinct leaves of $\mathcal{L}_s$ are mapped to distinct points in $\mathcal{T}$. The $\mathbb{R}$-tree $\mathcal{T}$ is equivalently obtained by collapsing each leaf of $\mathcal{F}_s$ to a point. A stabilizers of a branch-point is either conjugate to a subgroup associated to a subsurface S_i, $i > 0$, or is a cyclic subgroup corresponding to a boundary curve of S or to a reduction curve.

Proof: The first two assertions are an easy consequence of the fact that:

- $\tilde{h}$ is contracting by λ_0 along the leaves of $\mathcal{L}_s$,
- given any path p transverse to $\mathcal{F}_s$, some iterate of $\tilde{h}$ dilates the length of p by $C\lambda_0$, for some uniform constant C.

The last two assertions of the claim are clear.

Let us check the fourth assertion of the theorem. First we map the boundary of each conjugate of each subgroup associated to a subsurface S_i to the branch-point stabilized by this subgroup. Second, since the BBT-property of [34] is satisfied, there is a well-defined G-equivariant map from a subset of ∂G onto $\partial\mathcal{T}$ which is one-to-one. Thus the LL-map $\mathcal{Q}$ is well-defined on points $X \in \partial G$ which are limits of sequences $(g_n) \subset G$ whose translation-lengths on $\mathcal{T}$ tend to infinity. Choose a point $P \in \overline{\mathcal{T}}$ and let us now consider a sequence (g_nP) with $(g_n) \to X \in \partial G$ such that the translation-lengths of the g_n's do not tend toward infinity. Then, it is again a consequence of the BBT property that $\liminf_{n\to\infty} g_nP$ does not depend on the sequence (g_n) tending toward X. Let us prove that it is independent from P. It suffices to check it for a sequence (g_n) the translation-lengths of which, denoted by (t_n), tend to 0. If (t_n) is eventually constant then for some N all the g_k's with $k \in N$ belong to the stabilizer of some branch-point B. Since arc stabilizers are trivial, the g_kP turn around B, which is then equal to $\liminf_{n\to\infty} g_nP$. The same is true if the g_n's tend to some point in the boundary of the stabilizer of a branch-point. In the remaining case, the g_n's correspond to elements of G which are realized by a sequence of paths (p_n) closer and closer to a given leaf of $\mathcal{L}_s$. The terminal points of the p_n's belong to different leaves of $\mathcal{L}_s$. Hence this is true for the paths q_n associated to a sequence g_nq, where q is a fixed path from the base-point to a leaf in $\mathcal{L}_s$. Since different leaves of $\mathcal{L}_s$ correspond to different points of $\mathcal{T}$, the conclusion follows.

The continuity of the map $\mathcal{Q}$ is proved as in [19]. □

Remark 4.2. Assume that α is induced by a pseudo-Anosov homeomorphism h of a closed compact hyperbolic surface S. Let x, y be the endpoints in $\partial\mathbb{H}^2$ of the lift of a stable or unstable leaf of h. Then there are no geodesics in G_α from x to y: indeed the exponential contraction of the leaf implies that it can be exhausted by compact intervals I_n which are exponentially contracted by λ^{-t} for $t \to \infty$; this forbids the function $2t + \lambda^t|I_n|$ to attain a minimum for $|I_n| \to \infty$. In fact, the group G_α is hyperbolic and these two points of $\partial\mathbb{H}^2$ get identified to a single one in the Gromov compactification. In this case, $\mathcal{Q}(x)$ and $\mathcal{Q}(y)$ are the endpoints of a same eigenray, or the endpoints of two eigenrays with a same origin.

Definition 4.3. Let G be a discrete group acting by homotheties on an $\mathbb{R}$-tree $\mathcal{T}$. If R is an eigenray of some homothety H_g of $\mathcal{T}$, then *closing* R means identifying its two endpoints in $\widehat{\mathcal{T}}$.

Proposition 4.4. *With the assumptions and notations of Theorem 4.1, let $\widetilde{\mathcal{T}}$ be the space obtained by closing all the eigenrays in $\widehat{\mathcal{T}}$ and let $q\colon \widehat{\mathcal{T}} \to \widetilde{\mathcal{T}}$ be the associated quotient-map. Then:*

(1) $\widetilde{\mathcal{T}}$ is Hausdorff and compact.

(2) The action of G_α on $\widehat{\mathcal{T}}$ descend, via q, to an irreducible action by homeomorphisms on $\widetilde{\mathcal{T}}$. The induced G-action has dense orbits and restricts to an isometric action on the completion of $\mathcal{T} \subset \widetilde{\mathcal{T}}$. The G_α-action restricts to an affine action on the completion of $\mathcal{T} \subset \widetilde{\mathcal{T}}$.

(3) If X, Y are two distinct points in ∂G which cannot be connected by a bi-infinite G_α-geodesic, then $q(\mathcal{Q}(X)) = q(\mathcal{Q}(Y))$.

This result is certainly not a surprise to experts in the field of surface or free group automorphisms. In the case of mapping-tori of pseudo-Anosov surface homeomorphisms, it essentially amounts to the knowledge of the famous Cannon-Thurston map and of what are its fibers. We refer the reader to [16, 59] for this point of view. It seems however that the above statement has not been written under this form and under this generality, so that we give a brief proof for completeness.

Proof. The notations are those introduced in the proof of Theorem 4.1. Let l be a leaf of the unstable geodesic lamination $\mathcal{L}_u$ in $\hat{S}_{\mathcal{F}}$. Either it connects two points in $\partial\hat{S}_{\mathcal{F}}$, or it connects a point in $\partial\hat{S}_{\mathcal{F}}$ to the collapse of a lift of a reduction curve, a boundary curve or a subsurface S_i, $i > 0$. We now denote by l a leaf as above or the subset of a singular leaf of $\mathcal{F}_u$, homeomorphic to $\mathbb{R}$, the endpoints of which are of the above form. Such a subset is a *bad subset*. Then the endpoints of l are not connected by a G_α-geodesic: the argument given in Remark 4.2 in the pseudo-Anosov case applies here.

Conversely, any two points in $\hat{S}_{\mathcal{F}} \cup \partial\hat{S}_{\mathcal{F}}$ which are not the endpoints of a bad subset l as defined above are connected by a G_α geodesic. Indeed, they are connected by a path p which can be put transverse to the two foliations so that both its stable and unstable lengths (i.e. the lengths measured by integrating the stable and unstable measures of the foliations) are positive. It follows that sufficiently long subpaths of p are shortened under iterations of $\tilde{h}$ or $\tilde{h}^{-1}$ until reaching a positive minimal length L. The computations carried on in [39] imply that a G_α-geodesic passes in a neighborhood of the path with length L (the size of the neighbrohood depends on L which has been chosen sufficently large enough once and for all).

An eigenray R, with origin O and endpoint $w \in \partial\mathcal{T}$ satisfies $wH(R) = R$. Since w acts by isometries, whereas H is a homothety with dilatation factor $\lambda_0 > 1$, this implies that the length of any subpath in a geodesic g in $\hat{S}_{\mathcal{F}}$ from $\mathcal{Q}^{-1}(w)$ to the boundary of the convex-hull of $\mathcal{Q}^{-1}(O)$ is exponentially contracted under iterations of $\tilde{h}^{-1}$. Therefore the endpoints of g cannot be connected by a G_α-geodesic. It follows that its endpoints are the endpoints of a bad subset l of $\hat{S}_{\mathcal{F}} \cup \partial\hat{S}_{\mathcal{F}}$. Conversely it is clear that the endpoints of a bad subset l project, under $\mathcal{Q}$, to the endpoints of an eigenray or to the endpoints of two eigenrays with a same origin.

Now the endpoints of the bad subsets form a closed set in the following sense: if $(X_n, Y_n) \in \partial G \times \partial G$ are endpoints of bad subsets tending toward (X, Y) then X and Y are the endpoints of a bad subset. It follows that the collection of eigenrays satisfy a similar property: if R_n is a sequence of eigenrays with origin O_n and terminal point $w_n \in \partial\mathcal{T}$ with O_n (resp. w_n) tending toward some O (resp. w) in $\widehat{\mathcal{T}}$, then either O and w are the endpoints of an eigenray, or they are the endpoints of two eigenrays with same origin. It readily follows that closing the eigenrays gives a Hausdorff space. The compacity is then easily deduced from the compacity of $\widehat{\mathcal{T}}$.

We so got the first and third items of the proposition. The second item follows from the fact that G_α permutes the eigenrays. □

There are "generalizations" of these theorem and proposition in two directions: free groups and torsion free hyperbolic groups with infinitely many ends. The paper [34] gives the conclusions of Theorem 4.1 in the setting of free group automorphisms and the point (2) of Theorem 4.5 below dealing with polynomially growing automorphisms. The paper [56][Theorems 10.4 and 10.5] allows us to generalize [34] to torsion free hyperbolic groups with infinitely many ends. Let us recall that such a hyperbolic group G is the fundamental group of a graph of groups with trivial edge stabilizers and one-ended hyperbolic groups as vertex stabilizers. The *relative length* of an element γ of G is the word-length associated to the (infinite) generating set obtained by adding every element of any vertex stabilizer to the given generating set. An automorphism α of a torsion free hyperbolic group with infinitely many ends G has an *essential exponential growth* if there exists an element γ of G such that the relative length of $\alpha^j(\gamma)$ growths exponentially with $j \to +\infty$.

Theorem 4.5 ([35, 34, 54, 55, 56, 19])**.** *Let G be a free group or a torsion free hyperbolic group with infinitely many ends. Then:*

(1) The conclusions of Theorem 4.1 and Proposition 4.4 are true as soon as α is an automorphism with essential exponential growth.

(2) If α is polynomially growing then there is a minimal, simplicial α-invariant G-tree $\mathcal{T}$ which satisfies all the properties of Theorem 4.1.

Proof of Proposition 4.4 in the setting of Theorem 4.5. We consider the Cayley graph Γ of G with respect to some finite generating set. We consider a tree $\mathcal{T}$ as given by Theorem 4.5. Up to conjugacy in G, there are a finite number of stabilizers of branch-points $\mathcal{H} = \{H_1, \cdots, H_r\}$. The family $\mathcal{H}$ is quasi convex, and malnormal. We consider the coned-off Cayley graph $\Gamma_{\mathcal{H}}$ (see Section 9). We consider a cellular, piecewise linear map $f\colon \Gamma_{\mathcal{H}} \to \Gamma_{\mathcal{H}}$ which realizes the given automorphism. Since the subgroups H_i's are preserved up to conjugacy, f can be assumed to permutes the vertices of the cones in $\Gamma_{\mathcal{H}}$. We denote by $\Gamma^f_{\mathcal{H}}$ the mapping-telescope of $(\Gamma_{\mathcal{H}}, f)$. A *horizontal geodesic* in $\Gamma^f_{\mathcal{H}}$ is $\Gamma_{\mathcal{H}}$-geodesic contained in some *stratum* $\Gamma_{\mathcal{H}} \times \{j\}$, $j \in \mathbb{Z}$. It is *simple* if it contains no vertex of any cone in $\Gamma_{\mathcal{H}}$.

Definition 4.6. A *corridor* is a union of (possibly infinite or bi-infinite) simple horizontal geodesics, exactly one in each stratum, which connect two orbits of f.

A *K-quasi orbit* is a sequence of points $x_0, \cdots, x_j, \cdots$ such that there is a sequence of vertical segments $v_1, \cdots, v_{j+1}, \cdots$ of lengths at least one satisfying that the initial point of v_i is x_{i-1} and the terminal point of v_i lies at horizontal distance at most K from x_i. By [36], there is a constant K such that, if x is a point in a corridor $\mathcal{C}$, then there is a K-quasi orbit passing through x and contained in $\mathcal{C}$.

Definition 4.7. A corridor $\mathcal{C}$ is *collapsed toward* $+\infty$ (resp. *toward* $-\infty$) if, given any two points x, y in $\mathcal{C} \cap (\Gamma_{\mathcal{H}} \times \{j\})$, there exist $l \in \mathbb{N}$ and two K-quasi orbits starting respectively at x and y and ending at the same point in $\mathcal{C} \cap (\Gamma_{\mathcal{H}} \times \{j + l\})$ (resp. $\mathcal{C} \cap (\Gamma_{\mathcal{H}} \times \{j - l\})$).

A *collapsing corridor* is a corridor which collapses either toward $+\infty$ or toward $-\infty$.

The collapsing corridors play the rôle of the stable and unstable leaves in the setting of Theorem 4.1.

Lemma 4.8. *The subset of the collapsing corridors is closed in the following sens: if $\mathcal{C}$ is a non-collapsing corridor, then there exists $L > 0$ such that the horizontal L-neighborhood of $\mathcal{C}$ contains only non-collapsing corridors.*

Corollary 4.9. *Let $(X_n, Y_n) \subset \partial\Gamma_{\mathcal{H}} \times \partial\Gamma_{\mathcal{H}}$ be a sequence of points which are the endpoints of a sequence of collapsing corridors. Assume that $X_n \to X$ and $Y_n \to Y$. Then X and Y are the endpoints of a collapsing corridor.*

As in the setting of Theorem 4.1, two points in $\partial\Gamma_{\mathcal{H}}$ are connected by a G_α-geodesic if and only if they define a non-collapsing corridor. The correspondance between the collapsing corridors and the eigenrays is established as was previously done. The proof of the proposition is then completed in the same way. We leave the reader work out the details. $\square$

4.2. **Subgroups of polynomially growing free group automorphisms.**

Theorem 4.10. [11] *Let $\mathcal{P}$ be a finitely generated group and let $\theta\colon \mathcal{P} \to \mathrm{Out}(\mathbb{F}_n)$ be a monomorphism such that $\theta(\mathcal{P})$ consists entirely of polynomially growing automorphisms. Then there is a finite-index subgroup $\mathcal{U}$ of $\mathcal{P}$, termed* unipotent subgroup*, and a simplicial $\tilde{\theta}(\mathcal{U})$-invariant $\mathbb{F}_n$-tree $\mathcal{T}$ with $[\tilde{\theta}(\mathcal{U})] = \theta(\mathcal{U})$ which satisfies the following properties:*

(1) A vertex of $\mathcal{T}$ is fixed by all the isometries $H_{\tilde{\theta}(u)}$, $u \in \mathcal{U}$, and this is the unique fixed point of each one.
(2) The $\mathbb{F}_n$-action has trivial edge stabilizers.
(3) There is exactly one $\mathbb{F}_n$-orbit of edges.
(4) For each vertex v of $\mathcal{T}$, each $\mathrm{Stab}_{\mathbb{F}_n}(v)$-orbit of directions at v is invariant under the $\mathcal{U}$-action.

As was already observed, the map $\mathcal{Q}$ is well-defined here since vertex stabilizers are quasiconvex and arc stabilizers are trivial.

5. Cyclic extensions over exponentially growing automorphisms

The bulk of this section is to prove Theorem 5.1 below about cyclic extensions of surface groups over exponentially growing automorphisms. Then we will briefly show how the same methods apply to similar cyclic extensions of free groups and of torsion free hyperbolic groups with infinitely many ends, see Theorem 5.12.

5.1. **The surface case.**

Theorem 5.1. *Let G be the fundamental group of a compact, hyperbolic surface S (with or without boundary). Let $\alpha \in \mathrm{Aut}(G)$ be an exponentially growing automorphism in an outer-class induced by a homeomorphism of S. Let μ be a probability measure on $G_\alpha := G \rtimes_\alpha \mathbb{Z}$ whose support generates G_α as a semi-group.*

Then there exists an α-projectively invariant G-tree $\widehat{\mathcal{T}}$ with minimal interior such that, if $\widetilde{\mathcal{T}}$ denotes the space obtained by closing all the eigenrays of $\widehat{\mathcal{T}}$, then:

(1) $\mathbf{P}$-almost every sample path $\mathbf{x} = \{x_n\}$ converges to some $x_\infty \in \widetilde{\mathcal{T}}$.
(2) The hitting measure λ, which is the distribution of x_∞, is a non-atomic measure on $\widetilde{\mathcal{T}}$ such that $(\widetilde{\mathcal{T}}, \lambda)$ is a μ-boundary of (G_α, μ) and λ is the unique μ-stationary probability measure on $\widetilde{\mathcal{T}}$.

(3) *If the measure μ has finite first logarithmic moment and finite entropy with respect to a word-metric on G_α, then the measured space $(\widetilde{\mathcal{T}}, \lambda)$ is the Poisson boundary of (G_α, μ).*

We consider a $\mathbb{R}$-tree $\mathcal{T}$ as given by Theorem 4.1. We denote by t the generator of $\mathbb{Z}$ acting on the right as α on G. We choose as base-point O the fixed-point of the homothety H_α associated to the given automorphism α. Beware that t acts on the left as the homothety H_α^{-1} on $\widehat{\mathcal{T}}$.

Choose a branch-point O as base-point and identify G with a subset of the branch-points of $\mathcal{T}$ by considering the orbit $G.O$ of the base-point. Of course, since in general the branch-points have non-trivial stabilizers, infinitely many elements of G get identified with a single branch-point. However, the triviality of the arc-stabilizers implies that a same element of G cannot stabilize more than one point in $\mathcal{T}$, so that the above identification of G with the orbit of O makes sense. The following lemma is obvious:

Lemma 5.2. *If P is any point in $\widehat{\mathcal{T}}$, let $\mathcal{N}_{\widehat{\mathcal{T}}}(P)$ consist of a neighborhood $\mathcal{N}^{obs}(P)$ of P in $\widehat{\mathcal{T}}$ together with all the elements $gu \in G_\alpha$ such that $guO \in \mathcal{N}^{obs}(P)$ and $|gu| > N$ for some chosen positive N.*

Then $\widehat{\mathcal{T}}$, equipped with the above basis of neighborhoods $\mathcal{N}_{\widehat{\mathcal{T}}}(.)$, is a separable, compatible compactification of G_α.

The action of $G \lhd G_\alpha$ on $\widehat{\mathcal{T}}$ is the given isometric left action.

The action of $\mathbb{Z} < G_\alpha$ on $\widehat{\mathcal{T}}$ is given by $u.P = H_{u^{-1}}(P)$.

Unfortunately, the compactification won't satisfy the (CP) condition. We now need to close the eigenrays as in Theorem 5.1.

Proposition 5.3. *In the compactification of G_α by $\widehat{\mathcal{T}}$ given in Lemma 5.2, extend the quotient-map $q\colon \widehat{\mathcal{T}} \to \widetilde{\mathcal{T}}$ by the identity on G_α.*

Define $\mathcal{N}_{\widetilde{\mathcal{T}}}(P) = q(\mathcal{N}_{\widehat{\mathcal{T}}}(q^{-1}(P)))$.

Then $\widetilde{\mathcal{T}}$, equipped with the basis of neighborhoods $\mathcal{N}_{\widetilde{\mathcal{T}}}(.)$ is a separable, compatible compactification of G_α. The action of G_α on $\widetilde{\mathcal{T}}$ is irreducible.

Proof. By Proposition 4.4, $\widetilde{\mathcal{T}}$ is Hausdorff and compact. Since the quotient-map q restricts to the identity on $\mathcal{T}$, the neighborhood in G_α of a point of $\widetilde{\mathcal{T}}$ is the same as the neighborhood of this point in $\widehat{\mathcal{T}}$. It is then obvious that Proposition 5.3 gives a compatible compactification of G_α by $\widetilde{\mathcal{T}}$. The irreducibility of the action comes from Proposition 4.4. □

Let us see what happens with respect to the (CP) and (CS) conditions when considering this compactification.

Proposition 5.4. *The compactification of G_α with $\widetilde{\mathcal{T}}$ given by Proposition 5.3 satisfies Kaimanovich (CP) condition.*

Proof. Consider any sequence $(w_j) \in G$ which tends to some point $P \in \widehat{\mathcal{T}}$, which means that (w_jO) tends to P in $\widehat{\mathcal{T}}$. Let $v \in G \lhd G_\alpha$. By Theorem 4.1, the LL-map $\mathcal{Q}$ is well-defined. By Lemma 3.10, $(w_j.(vO)) = (w_jvO)$ tends to the same point $P \in \widehat{\mathcal{T}}$. The condition (CP) is thus satified for $\widehat{\mathcal{T}}$, and so for $\widetilde{\mathcal{T}}$, in this case. If $v = t^k$ for some integer k the same conclusion holds trivially from the very definition of the compactification: the elements w and wt^k lie in a neighborhood of the same point in the tree since the base-point O has been chosen as the fixed-point of H, that is of the action of t on $\widehat{\mathcal{T}}$.

Let us now consider a sequence t^{n_j} with $n_j \to +\infty$. Of course $t^{n_j}O$ tends to O in $\widehat{\mathcal{T}}$ so that t^{n_j} tends to O in the compactification with $\widetilde{\mathcal{T}}$. Let $v \in G_\alpha$. If $v = t^k$ for some integer k then $t^{n_j}v = t^{n_j+k}$ tends to O in $\widehat{\mathcal{T}}$ so that the (CP) condition still holds. If $v \in G$, $t^k vO$ admits O and the endpoints of the eigenrays of H as accumulation points. Thus it converges in $\widetilde{\mathcal{T}}$ to the point O, since the eigenrays have been closed. Therefore the (CP) condition holds in $\widetilde{\mathcal{T}}$ in this case.

The lemma readily follows. □

Remark 5.5. In the last case treated in the proof of Proposition 5.4, we really need to be in $\widetilde{\mathcal{T}}$, and not only in $\widehat{\mathcal{T}}$.

Definition 5.6. We identify ∂G with $\partial G \times \{0\}$. If $K \subset \partial G$ is compact, we denote by $\Lambda(K)$ the convex-hull of K in $G \cup \partial G$.

If K_1, K_2 are two compact subsets of ∂G, we denote by $\mathcal{G}(K_1, K_2)$ the union of all the G_α-geodesics between $\partial\Lambda(K_1) \subset \partial G$ and K_2.

Let Δ be the diagonal of $\widehat{\mathcal{T}} \times \widehat{\mathcal{T}}$. Let $\mathcal{R}$ be a set of representatives of G_α-orbits in $(\widehat{\mathcal{T}} \times \widehat{\mathcal{T}} \setminus \Delta)/((x,y) \sim (y,x))$.

If $(b_1'', b_2'') \in \mathcal{R}$, $ES'(b_1'', b_2'') = \mathcal{G}(\mathcal{Q}^{-1}(b_1''), \mathcal{Q}^{-1}(b_2'')) \cup \mathcal{G}(\mathcal{Q}^{-1}(b_2''), \mathcal{Q}^{-1}(b_1''))$.

If $(b_1', b_2') \in \widehat{\mathcal{T}} \times \widehat{\mathcal{T}} \setminus \Delta$, the *elementary strip* $ES(b_1', b_2')$ *between* b_1' *and* b_2' is the union of all the $gu.ES'(b_1'', b_2'')$ with $gu\{b_1'', b_2''\} = \{b_1', b_2'\}$.

Let $(b_1, b_2) \in \widetilde{\mathcal{T}} \times \widetilde{\mathcal{T}} \setminus \Delta$. The *strip* $S(b_1, b_2)$ is defined by

$S(b_1, b_2) = ES(b_1', b_2')$ with $b_i' = b_i$ if $q^{-1}(b_i) = b_i$ and b_i' is the unique origin of eigenray in $q^{-1}(b_i)$ otherwise.

Lemma 5.7. *No strip is empty.*

Proof. Let b_1, b_2 be two distinct points in $\widetilde{\mathcal{T}}$. We denote by b_i' the points in $\widehat{\mathcal{T}}$ with $q(b_i') = b_i$ given in Definition 5.6 for defining $S(b_1, b_2)$. Let K_i, $i = 1, 2$, be the convex-hulls in $G \cup \partial G$ of $\mathcal{Q}^{-1}(b_i')$. If there is a non-collapsing corridor between ∂K_i and K_j then the strip is non-empty. From Lemma 4.8, a collapsing corridor occurs if and only if b_1' and b_2' are the endpoints of an eigenray. Since the eigenrays have been closed in $\widetilde{\mathcal{T}}$, we would get $b_1 = b_2$ which is a contradiction with the assumption. □

Lemma 5.8. *The map*

$$\begin{cases} \widetilde{\mathcal{T}} & \times & \widetilde{\mathcal{T}} & \to & G_\alpha \\ (b_1 & , & b_2) & \mapsto & S(b_1, b_2) \end{cases}$$

is G_α-equivariant and Borel.

Proof. The G_α-equivariance is clear by construction. We only have to check that the map which assigns the elementary sets is Borel. This is straightforward since two distinct points in $\widetilde{\mathcal{T}} \times \widetilde{\mathcal{T}} \setminus \Delta$ have distinct image sets. Therefore any set in G_α, which is countable, is a countable union of points in $\widetilde{\mathcal{T}} \times \widetilde{\mathcal{T}} \setminus \Delta_{\widetilde{\mathcal{T}}}$, hence is Borel. □

Proposition 5.9. *The G_α-compactification $\widetilde{\mathcal{T}}$, equipped with the collection of strips $S(b_1, b_2)$, satisfies the (CS) condition.*

Proof. From Lemma 5.7, no strip is empty. From Lemma 5.8, the assignment of the strips is Borel and G_α-equivariant. It only remains to check that any strip $S(b_1, b_2)$ avoids a neighborhood of any third point $b_0 \in \widetilde{\mathcal{T}}$ distinct from both b_1 and b_2. If no element of G_α fixes both b_1 and b_2, then $S(b_1, b_2)$ consists of a finite set of G_α-geodesics. The accumulation-points of these geodesics are by definition the boundary-points. If some element wt^j would fix both b_1 and b_2, they would be the endpoints of an eigenray. Since eigenrays have been closed in $\widetilde{\mathcal{T}}$, if b_1 and b_2 are fixed by a same element of G_α this is an element w in G which acts as a hyperbolic isometry whose axis admits b_1 and b_2 as endpoint. It readily follows that these are the only accumulation-points. $\square$

Proposition 5.10. *Let $\mathcal{J}$ be a finite word gauge for G_α. Then the strips $S(b_1, b_2)$ given by Proposition 5.9 grow polynomially with respect to $\mathcal{J}$.*

Proof. The group G_α is strongly hyperbolic relative to the mapping-tori of the maximal subgroups where the automorphism α has linear growth (see Section 9, Theorem 9.4). Outside the incompressible tori bounding these submanifolds, the G_α-geodesics behave like in a hyperbolic group [26, 61]. Thus the number of intersection-points of the strip with a ball of region k growths at most linearly with k outside the mapping-tori of the α-polynomial growth subgroups. Inside these mapping-tori, it is easily proved that this same number is bounded above by a polynomial of degree 3. Indeed on the one hand the α-growth of an element is bounded above by a polynomial of degree 2. On the other hand the G_α-geodesics remain in a bounded neighborhood of a corridor between the orbits of their entrance- and exit-points. Thus the bound is given by the product of a linear map with a degree 2 polynomial. $\square$

Proof of Theorem 5.1. We consider a G-tree $\mathcal{T}$ as given by Theorem 4.1. By Propositions 5.3, 5.9, 5.4, Theorem 1.3 applies and gives the first point of Theorem 5.1. Proposition 5.10 and Theorem 1.3 give the Poisson boundary. $\square$

Corollary 5.11. *With the assumptions and notations of Theorem 5.1:*

(1) There is a topology on $G_\alpha \cup \partial G$ such that $\mathbf{P}$-almost every sample path $\mathbf{x} = \{x_n\}$ of the random walk converges to some $x_\infty \in \partial G$.

(2) The hitting measure λ (i.e. the distribution of x_∞) is a non-atomic measure on ∂G such that $(\partial G, \lambda)$ is a μ-boundary of (G_α, μ), and this is the unique μ-stationary measure on ∂G.

(3) If μ has finite first logarithmic moment and finite entropy with respect to some finite word-metric on G_α, then the measured space $(\partial G, \lambda)$ is the Poisson boundary of (G_α, μ).

Proof. The map $q \circ \mathcal{Q}\colon \partial G \to \widetilde{\mathcal{T}}$ is continuous, surjective and G_α-equivariant. Item (1) of Theorem 5.1 then gives Item (1) of the current corollary. The map $q \circ \mathcal{Q}$ is a continuous projection such that the disjoint sets that get identified to a single point are the disjoint translates of a finite number of disjoint compact subsets of ∂G. We can thus define a hitting measure λ on ∂G such that $(q \circ \mathcal{Q})_* \lambda$ is the μ-stationary measure on $\widetilde{\mathcal{T}}$ given by Theorem 5.1 (the pre-image of a point has λ-measure 0). The non-atomicity, μ-stationarity and unicity of λ follow from the non-atomicity, μ-stationarity and the unicity of the measure on $\widetilde{\mathcal{T}}$ given by Theorem 5.1. We so got Item (2), and Item (3) is a straightforward consequence of Item (3) of Theorem 5.1. $\square$

5.2. **Generalizations to free and hyperbolic groups.** The following result is now given by Theorem 4.5.

Theorem 5.12. *Theorem 5.1 and Corollary 5.11 remain true*

- *either if one substitutes a free group to the fundamental group of a compact hyperbolic surface,*

- *or if one substitutes a torsion free hyperbolic group with infinitely many ends G to the fundamental group of a compact hyperbolic surface and the automorphism has essential exponential growth.*

Observe that, although G has infinitely many ends, the cyclic extension $G \rtimes_\alpha \mathbb{Z}$ is one-ended.

6. Cyclic extensions over polynomially growing automorphisms

The core of this section is to prove Theorem 6.2 below. Applications to particular classes of groups follow, see Theorem 6.23.

Definition 6.1. Let $\mathcal{T}$ be a $\mathbb{R}$-tree. *Closing a bi-infinite geodesic* in $\mathcal{T}$ means identifying its endpoints in $\partial\mathcal{T}$.

Theorem 6.2. *Let G be a hyperbolic group. Let $\alpha \in \mathrm{Aut}(G)$ be a polynomially growing automorphism. Assume the existence of a simplicial α-invariant G-tree $\mathcal{T}$ such that:*

- *The G-action has quasi convex vertex stabilizers, trivial edge stabilizers and only one orbit of edges.*
- *The α-action fixes exactly one vertex O and fixes each $\mathrm{Stab}_G(O)$-orbit of directions at O.*

Let μ be a probability measure on $\mathcal{G} = G \rtimes_\alpha \mathbb{Z}$ whose support generates $\mathcal{G}$ as a semi-group. Then, if $\widetilde{\mathcal{T}}$ is obtained from $\widehat{\mathcal{T}}$ by closing each geodesic in the G-orbit of some bi-infinite geodesic:

1. *There is a topology on $\mathcal{G} \cup \partial\widetilde{\mathcal{T}}$ such that $\mathbf{P}$-almost every sample path $\mathbf{x} = \{x_n\}$ of the random walk converges to some $x_\infty \in \partial\widetilde{\mathcal{T}}$.*
2. *The hitting measure λ is a non-atomic measure on $\partial\widetilde{\mathcal{T}}$ such that $(\partial\widetilde{\mathcal{T}}, \lambda)$ is a μ-boundary of $(\mathcal{G}, \mu)$, and this is the unique μ-stationary measure on $\partial\widetilde{\mathcal{T}}$.*
3. *If μ has finite first logarithmic moment and finite entropy with respect to some finite word-metric on $\mathcal{G}$ then the measured space $(\partial\widetilde{\mathcal{T}}, \lambda)$ is the Poisson boundary of $(\mathcal{G}, \mu)$.*

An important intermediate step is to first prove Theorem 6.4 below. However we need an additional notion before its statement:

Definition 6.3. Let $\Phi \in \mathrm{Out}(G)$ and let $\mathcal{T}$ be a Φ-invariant G-tree. Let $\alpha, \beta \in \mathrm{Aut}(G)$ with $[\alpha] = [\beta] = \Phi$. We denote by H_α (resp. H_β) the corresponding isometries of $\mathcal{T}$, i.e. for any $w \in G$ and $P \in \mathcal{T}$, $H_\alpha(wP) = \alpha(w)H_\alpha(P)$ and $H_\beta(wP) = \beta(w)H_\beta(P)$.

The *(α, β)-action of $\mathcal{G} = G \times_\Phi \mathbb{Z}$ on $\mathcal{T} \times \mathcal{T}$* is the action given by $\Theta\colon \mathcal{G} \to \mathrm{Isom}(\mathcal{T} \times \mathcal{T})$ with:

$$\Theta(wt) \left\{ \begin{array}{ccccccc} \mathcal{T} & \times & \mathcal{T} & \to & \mathcal{T} & \times & \mathcal{T} \\ (P & , & Q) & \mapsto & (wH_\alpha^{-1}(P) & , & wH_\beta^{-1}(Q)) \end{array} \right.$$

Theorem 6.4. *Let G be a hyperbolic group. Let $\alpha \in \mathrm{Aut}(G)$ be a polynomially growing automorphism. Assume the existence of a simplicial α-invariant G-tree $\mathcal{T}$ such that:*

- *The G-action has quasi convex vertex stabilizers, trivial edge stabilizers and only one orbit of edges.*
- *The α-action fixes exactly one vertex O and fixes each $\mathrm{Stab}_G(O)$-orbit of directions at O.*

Let μ be a probability measure on $\mathcal{G} = G \rtimes_\alpha \mathbb{Z}$. Then there exist $\beta \in \mathrm{Aut}(G)$ with $[\alpha] = [\beta]$ in $\mathrm{Out}(G)$, and a bi-infinite geodesic A such that, if $\widetilde{\mathcal{T}}$ is obtained from $\widehat{\mathcal{T}}$ by closing each geodesic in the G-orbit of A and $\partial\overline{\mathcal{G}.O}$ is the boundary of the closure of the (α, β)-orbit of O in $\partial(\widetilde{\mathcal{T}} \times \widetilde{\mathcal{T}})$ then:

(1) $\mathbf{P}$-almost every sample path $\mathbf{x} = \{x_n\}$ converges to some $x_\infty \in \partial\overline{\mathcal{G}.O}$.

(2) The hitting measure λ, which is the distribution of x_∞, is a non-atomic measure on $\partial\overline{\mathcal{G}.O}$ such that $(\partial\overline{\mathcal{G}.O}, \lambda)$ is a μ-boundary of $(\mathcal{G}, \mu)$ and λ is the unique μ-stationary probability measure on $\partial\overline{\mathcal{G}.O}$.

(3) If the measure μ has finite first logarithmic moment and finite entropy with respect to a word-metric on $\mathcal{G}$, then the measured space $(\partial\overline{\mathcal{G}.O}, \lambda)$ is the Poisson boundary of $(\mathcal{G}, \mu)$.

We set $G = \langle x_1, \cdots, x_n \rangle$. The generator t of $\mathbb{Z}$ acts on the left as the isometry H^{-1} with H the isometry of $\mathcal{T}$ satisfying $H(wP) = \alpha(w)H(P)$. Let us observe that the simplicial nature of $\mathcal{T}$ gives us an easy identification of G with the orbit of some vertex.

Lemma 6.5. *Choose the point O given in Theorem 6.4 at base-point and if there are two orbits of vertices, choose another base-point O' adjacent to O. Then any vertex x of $\mathcal{T}$ is associated to a unique left-class wH_i $(i = 1, 2)$ as follows:*

- *w is the element of G associated to the unique geodesic from O to x.*
- *H_i is the stabilizer of the base-point which belongs to the orbit of x.*

Equipping $\mathcal{T}$ with the observers topology, and after identifying G_α with the orbit of O, we get a compatible compactification of G_α in a way similar to Lemma 5.2. This compactification however does not necessarily satisfy the (CP) property: indeed it might happen that some isometry vt^j, $v \in G$, fixes more than one point. This would imply that, for some $w \in G$, the sequences $\{(vt^j)^k\}_{k=1,\cdots,+\infty}$ and $\{(vt^j)^k w\}_{k=1,\cdots,+\infty}$ would not have the same limit-point.

By assumption, t fixes each $\mathrm{Stab}_G(x)$-orbit of direction at O. Thus, there exists an isometry of G_α fixing more than one vertex of $\mathcal{T}$, it has the form vt.

The important observation holds in the following well-known observation:

Lemma 6.6. *If both vt and wt^k fix more than one vertex in $\mathcal{T}$ then there is $g \in G$ with $(vt)^k = g^{-1}wt^k g$. If vt and wt^k fix the same edge, then $w = v\alpha^{-1}(v) \cdots \alpha^{1-k}(v)$.*

Proof. Assume that vt and wt both fix at least two vertices. Since there is only one G-orbit of edges, without loss of generality we can assume that vt and a G-conjugate of wt, denoted by $g^{-1}wtg$, both fix the same edge E. Then $vtg^{-1}t^{-1}w^{-1}g = w\alpha^{-1}(g^{-1})w^{-1}g$ fixes E. By the triviality of the edge stabilizers, $vtg^{-1}t^{-1}w^{-1}g$ is trivial. We so get the lemma in the case $k = 1$, the generalization is straightforward. □

Definition 6.7. A *singular element* for an action of G_α on $\mathcal{T}$ is an element of G_α which fixes at least two vertices of $\mathcal{T}$.

If vt is a singular element, the trick now is to consider another action of G_α on $\mathcal{T}$ by making t act on $\mathcal{T}$ by another automorphism in the same outer-class so that vt acts on this copy of $\mathcal{T}$ as a hyperbolic element.

Lemma 6.8. *With the notations above: there is an (α, β)-action of $\mathcal{G}$ on $\mathcal{T} \times \mathcal{T}$ such that any element of $\mathcal{G}$ which is a singular element for the action induced on one of the two factors ($\mathcal{T} \times \{*\}$ or $\{*\} \times \mathcal{T}$) acts as a hyperbolic isometry on the other factor and the axis of this hyperbolic isometry either is disjoint from the fixed set of the singular element, or is an axis in this fixed set.*

The axis of this hyperbolic isometry is called a singular axis*.*

Proof. Assume that vt is a singular element. Let T be the tree fixed by vt. We choose a hyperbolic element a of G such that avt acts as an hyperbolic isometry on $\mathcal{T}$, the axis of which is disjoint from T if $T \neq \mathcal{T}$. We consider the (α,β)-action of $\mathcal{G}$ given by:

$$\Theta(wt) \left\{ \begin{array}{ccccccc} \mathcal{T} & \times & \mathcal{T} & \to & \mathcal{T} & \times & \mathcal{T} \\ (P & , & Q) & \mapsto & (wH^{-1}(P) & , & wv^{-1}avH^{-1}(Q)) \end{array} \right.$$

If wt is a singular element for the first factor, then by Lemma 6.6, there exists $g \in G$ with $wt = g^{-1}vtg$. Since vt acts on the second factor as the hyperbolic isometry avt, wt acts on this second factor as $g^{-1}avtg$, which is also an hyperbolic isometry.

Assume now that wt acts as a singular element on the second factor. Since vt acts on the second factor like $vv^{-1}avt = avt$ then $a^{-1}vt$ is a singular element for the action on the second factor. Thus wt acts on the first factor as a conjugate to $a^{-1}vt$. Since avt is a hyperbolic isometry of $\mathcal{T}$, so is $a^{-1}vt$ and so is any of its conjugates. □

Of course any (α,β)-action extends to an action on $\widehat{\mathcal{T}} \times \widehat{\mathcal{T}}$.

Definition 6.9. A *singular boundary-tree* of $\mathcal{T} \times \mathcal{T}$ is a product $\partial A \times T$ or $T \times \partial A$ where:

- A is a singular axis.
- T is the closure in $\mathcal{T} \cup \partial\mathcal{T}$ of a maximal subtree of $\mathcal{T}$ which is fixed by the singular element of singular axis A.

Definition 6.10. We denote by:

- $\widetilde{\mathcal{T}}$ the space obtained from $\widehat{\mathcal{T}}$ by closing all the singular axis.
- $\widetilde{\mathcal{T}}^2$ the space obtained from $\widetilde{\mathcal{T}} \times \widetilde{\mathcal{T}}$ by identifying each singular boundary-tree to a point.

Lemma 6.11. *The space $\widetilde{\mathcal{T}}^2$ is Hausdorff and compact. The (α,β)-action of $\mathcal{G}$ on $\mathcal{T} \times \mathcal{T}$ induces an irreducible action on $\widetilde{\mathcal{T}}^2$.*

Proof. The tree $\mathcal{T}$ is simplicial thus separable. By Proposition 3.3 $\widehat{\mathcal{T}}$ is Hausdorff and compact. By construction, there is an axis A of $\mathcal{T}$ such that $\widetilde{\mathcal{T}}$ is obtained from $\widehat{\mathcal{T}}$ by identifying two ends in $\partial\mathcal{T}$ if and only if there are the ends of an axis gA, $g \in G$. This is easily seen to be a closed property in $\partial\mathcal{T} \times \partial\mathcal{T}$. It readily follows that $\widetilde{\mathcal{T}}$ is Hausdorff and compact. By definition of a singular axis, there is an element w of G such that $t.A = w.A$. It follows that the whole orbit $G.A$ is invariant under the $\mathcal{G}$-action. Thus the $\mathcal{G}$-action induces an action on $\widetilde{\mathcal{T}}$. This action is irreducible because it was on $\widehat{\mathcal{T}}$. It readily follows that $\widetilde{\mathcal{T}} \times \widetilde{\mathcal{T}}$ is Hausdorff, compact and the (α,β)-action of $\mathcal{G}$ is an irreducible action. From Lemma 6.16, to pass to $\widetilde{\mathcal{T}}^2$, one identifies to a point each singular boundary-tree in a single G-orbit. The arguments for completing the proof are then the same as the arguments to pass from $\widehat{\mathcal{T}}$ to $\widetilde{\mathcal{T}}$. □

Although important, the following proposition is however obvious from Lemma 6.11:

Proposition 6.12. *Let $O \in \widetilde{\mathcal{T}}^2$ be the point whose coordinates are the fixed-point of α.*

If P is any point in $\widetilde{\mathcal{T}}^2$, let $\mathcal{N}(P)$ consist of a neighborhood $\mathcal{N}^{obs}(P)$ of P in $\widetilde{\mathcal{T}}^2$ equipped with the product topology, together with all the elements $wt^k \in G_\alpha$ such that $\Theta(wt^k)(O) \in \mathcal{N}^{obs}(P)$ and $|wt^k| > N$ for some chosen positive N.

Then $\widetilde{\mathcal{T}}^2$, equipped with the above basis of neighborhoods $\mathcal{N}(.)$, is a separable, compatible compactification of the group G_α.

Proposition 6.13. *The compactification of $\mathcal{G}$ by $\overline{\mathcal{G}.O} \subset \widetilde{\mathcal{T}}^2$ satisfies the (CP) condition.*

Before proving Proposition 6.13, we study with more details the action of $\mathcal{G}$ on $\widetilde{\mathcal{T}}^2$. Even if we won't need the full strength of the two lemmas below, they might be useful to help the reader having a better grasp on what happens here.

Definition 6.14. A *rectangle* in $\mathcal{T} \times \mathcal{T}$ is a product of two geodesics.

A *singular rectangle* is a rectangle which is the product of a singular axis with a geodesic.

A *corner* of a rectangle R is a point in $\partial R \cap (\partial\mathcal{T} \times \partial\mathcal{T})$, where $\partial R = \overline{R} \setminus R$ in $\widehat{\mathcal{T}} \times \widehat{\mathcal{T}}$.

Lemma 6.15. *We consider the (α, β)-action of $\mathcal{G}$ on $\mathcal{T} \times \mathcal{T}$ given in Lemma 6.8.*

Let $R = g_1 \times g_2$ be a singular rectangle the stabilizer of which is neither trivial nor cyclic. Then $g_1 = g_2$ and, up to taking powers, there are a unique $v \in G$ and $wt \in G_\alpha$ such that v admits g_1 as hyperbolic axis and g_1 is a singular axis for the action of wt. In particular, the actions of v and wt commute and the stabilizer of R is a $\mathbb{Z} \oplus \mathbb{Z}$-subgroup.

Proof. Since two distinct G-elements cannot share the same axis in $\mathcal{T}$, there is at most one $v \in G$, up to taking powers, admitting both g_1 and g_2 as hyperbolic axis. Since the action of G on $\mathcal{T} \times \mathcal{T}$ is just the diagonal action, if there is one v fixing R then $g_1 = g_2$.

Let us now prove that there is at most one element in $\mathcal{G}$, which does not belong to G and admit both g_1 and g_2 as hyperbolic axis. If there were two then, up to taking powers, we can assume that they have the form vt^k and wt^k. Then $vw^{-1} = vt^k t^{-k} w^{-1}$ fixes the same axis. This is an element of G and, since $\mathcal{T}$ is a α-invariant tree, vw^{-1} is fixed by the automorphisms $v\alpha^k(.)v^{-1}$ and $w\alpha^k(.)w^{-1}$. Therefore $\alpha^k(vw^{-1}) = w^{-1}v\alpha^k(vw^{-1})v^{-1}w$. Since G is hyperbolic, vw^{-1} is then either a torsion element, or trivial. Since it fixes a point in ∂T, and edge stabilizers are trivial, it cannot be a torsion element. Thus $vw^{-1} = 1_G$. We so got that, up to taking powers, there is at most one vt fixing R.

From the previous two paragraphs, if R is a singular rectangle whose stabilizer is neither trivial nor cyclic, then up to taking powers there are a unique $v \in G$ and $wt \in \mathcal{G}$ fixing R. Since t does not act in the same way on the two factors, if wt fixes R, then one of the two axis is a singular axis for wt. Thus v and wt commute and we get the lemma. ☐

Lemma 6.16. *We consider the (α, β)-action of $\mathcal{G}$ on $\mathcal{T} \times \mathcal{T}$ given in Lemma 6.8.*

1. *There is at most one G-orbit of singular rectangles the stabilizers of which are neither trivial nor cyclic.*
2. *Let R be a rectangle with cyclic stabilizer $\langle w_1 t^{n_1} \rangle$. For any point $P \in \widehat{\mathcal{T}} \times \widehat{\mathcal{T}}$, the accumulation points of $\Theta(\langle w_1 t^{n_1} \rangle)(P)$ either are two opposite corners of R, or belong to a singular boundary-tree.*

Proof. Item (1) comes directly from Lemmas 6.6 (unicity of the singular element) and 6.15. Consider now a rectangle R with cyclic stabilizer in G. If this cyclic stabilizer is not generated by a singular element, we are in the case where the orbit of O accumulates on two opposite corners. If the cyclic stabilizer is generated by a singular element, we are in the second case. ☐

Proof of Proposition 6.13. Consider a sequence $v_i t^{n_i}$ such that $\Theta(v_i t^{n_i}).O$ converges to some point P. If the points of the sequence belong to an infinite number of rectangles then this is also true for any sequence $\Theta(v_i t^{n_i} w).O$, $w \in G$. The convergence of $\Theta(v_i t^{n_i}).O$ to P means that $\Theta(v_i t^{n_i}).O$ turns around P, and so does $\Theta(v_i t^{n_i} w).O$. Let us assume that there exists $N \geq 0$ such that for all $i \geq N$, all the $\Theta(v_i t^{n_i}).O$ belong to a same rectangle. Then again either the points turn around P and the conclusion for $\Theta(v_i t^{n_i} w).O$ is straightforward. Or the rectangle has a non-trivial stabilizer and the conclusion for

$\Theta(v_i t^{n_i} w).O$ is deduced from Lemma 6.16 since each singular boundary-tree has been collapsed to a point. We conclude by noticing that, since t fixes O, the above arguments readily imply the (CP) condition. $\square$

Definition 6.17. With the assumptions and notations of Lemma 6.5:

The *cylinder of a left-class* wH_i is the set of all elements $wH_it \in G_\alpha$. *A cylinder* is the cylinder of some left-class wH_i.

Let g be a $\mathcal{G}$-geodesic and let $\mathcal{C}$ be a cylinder. If $w_0t^{k_0}w_1t^{k_1}\cdots w_it^{k_i}w_{i+1}t^{k_{i+1}}$ are consecutive elements in g such that for all $1 \leq j \leq i$, the $w_jt^{k_j}$ belong to $\mathcal{C}$ whereas $w_0t^{k_0}$ and $w_{i+1}t^{k_{i+1}}$ do not, then $w_1t^{k_1}$ (resp. $w_it^{k_i}$) is an *entrance-point* (resp. an *exit-point*) of g in $\mathcal{C}$.

Let b_1, b_2 be any two distinct points in $\widehat{\mathcal{T}}$. The *basic strip* $BS(b_1, b_2)$ consists of all the entrance and exit-points of the G_α-geodesics between b_1 and b_2 in the various cylinders they pass through.

Definition 6.18. Let $\mathcal{R}$ be a set of representatives of $\mathcal{G}$-orbits in $(\overline{\mathcal{G}.O}\times\overline{\mathcal{G}.O}\setminus\Delta)/((x,y)\sim(y,x))$.

If (b_1', c_1') and (b_2', c_2') are two distinct elements in $\mathcal{R}$, the *elementary strip* $ES(b_1', c_1', b_2', c_2')$ is equal to $BS(b_1', b_2') \cup BS(c_1', c_2')$.

Let (b_1, c_1) and (b_2, c_2) be two distinct elements in $\overline{\Theta(G_\alpha).O}$. The *strip* $S(b_1, c_1, b_2, c_2)$ is the union of all the $w_it^{k_i}ES(b_1^i, c_1^i, b_2^i, c_2^i)$ such that $w_it^{k_i}.\{(b_1^i, c_1^i), (b_2^i, c_2^i)\} = \{(b_1, c_1), (b_2, c_2)\}$.

Proposition 6.19. *The compactification of* $\mathcal{G}$ *by* $\overline{\mathcal{G}.O} \subset \widetilde{\mathcal{T}}^2$*, equipped with the collection of strips* $S(b_1, b_2)$*, satisfies the (CS) condition.*

Proof. Since the bi-infinite horizontal geodesic between two points in $\partial\mathcal{T}$ is sent isometrically to another such bi-infinite by any element in $\mathcal{G}$, no basic strip is empty. Thus no strip is empty. The $\mathcal{G}$-equivariance is clear by construction. It remains to check that the strip $S(b_1, c_1, b_2, c_2)$ does not accumulate on a third boundary point b_0. If $\{(b_1, c_1), (b_2, c_2)\}$ is not stabilized by a G_α-element the strip is contained in a finite number of G_α-geodesics, connecting b_1 to b_2 and c_1 to c_2. It readily follows that the only accumulation points are (b_1, c_1) and (b_2, c_2). If $\{(b_1, c_1), (b_2, c_2)\}$ is stabilized, then they are the two opposite corners of a rectangle with cyclic stabilizer, since the endpoints of the singular axis have been identified and the boundaries of the singular rectangle with non-trivial nor cyclic stabilizer have been identified to a point. This cyclic stabilizer acts like a hyperbolic isometry with endpoints the two opposite corners. Hence the conclusion and the lemma. $\square$

Proposition 6.20. *Let* $\mathcal{J}$ *be a finite word gauge for* $\mathcal{G}$*. Then the strips* $S(b_1, b_2)$ *given by Proposition 6.19 grow polynomially with respect to* $\mathcal{J}$*.*

Proof. There are three possibilities for the considered strip $S(b_1, c_1, b_2, c_2)$:

- It is contained in a finite number of G_α-geodesics.
- The two points (b_1, c_1) and (b_2, c_2) are stabilized by an element of G_α not in G.
- The two points (b_1, c_1) and (b_2, c_2) are stabilized by an element of G.

The inequality is straightforward in the first case. In the second case, it suffices to observe that, for any $G \times \{k\}$, there are only a finite number of G_α-geodesics in the strip which intersect $G \times \{k\}$. In the third case, the various entrance- and exit-points the various $\mathcal{G}$-geodesics in the strip pass through are permuted under the translation of the element of G considered. Thus this case follows from the first one. $\square$

Proof of Theorem 6.4. Proposition 6.12 gives that $\overline{\mathcal{G}.O} \subset \widetilde{\mathcal{T}}^2$ is a separable, compatible compactification of $\mathcal{G}$ with an irreductible action. Propositions 6.13 and 6.19 give

Kaimanovich (CP) and (CS) properties. By Theorem 1.3, we so get the conclusions of the first point of Theorem 6.4 for the union of $\partial\overline{\mathcal{G}.O} \subset \partial\widetilde{\mathcal{T}}^2$ with the vertices of $\mathcal{T} \times \mathcal{T}$ in $\overline{\mathcal{G}.O} \setminus \mathcal{G}.O$. Since this set of vertices is invariant, and the measure λ non-atomic, it has λ-measure zero, which gives to us Theorem 6.4, Item (2). Theorem 1.3 and Proposition 6.20 give the third point. $\square$

Proof of Theorem 6.2. We call *singular points* the points resulting from the collapsing of the singular boundary-trees. There is a slight abuse of terminology in the lemma below when considering the "projection on the first factor" $\pi\colon \partial\widetilde{\mathcal{T}}^2 \setminus \{\text{singular points}\} \to \widetilde{\mathcal{T}}$. We mean of course the map induced by the projection on the first factor from $\partial(\widetilde{\mathcal{T}} \times \widetilde{\mathcal{T}})$ to $\widetilde{\mathcal{T}}$.

Lemma 6.21. *Let $\pi\colon \partial\widetilde{\mathcal{T}}^2 \setminus \{\text{singular points}\} \to \widetilde{\mathcal{T}}$ be the projection on the first factor. Then:*

(1) For any $x \in \partial\widetilde{\mathcal{T}}$ lying in the image of π, $\pi^{-1}(x) \cap \partial\overline{\mathcal{G}.O}$ consists of at most one point.

(2) If $V(\mathcal{T})$ denotes the set of vertices of $\mathcal{T}$, then there exist either one or two vertices x_0, x_1 of $\mathcal{T}$ such that $\{\pi^{-1}(x), x \in V(\mathcal{T})\} = \mathcal{G}.\pi^{-1}(x_0) \sqcup \mathcal{G}\pi^{-1}(x_1)$ and all the translates $\gamma.\pi^{-1}(x_0)$ and $\gamma.\pi^{-1}(x_1)$,$\gamma \in \mathcal{G}$, are disjoint as soon as the translating element γ is not in the stabilizer.

(3) The map π is $\mathcal{G}$-equivariant.

Proof. Items (2) and (3) are clear. Item (1) is a consequence of the fact that the points in $\partial\mathcal{T}$ which have neither trivial nor cyclic stabilizer are the endpoints of singular axis. Indeed, if v and wt^k both fix $P \in \partial\mathcal{T}$ then wt^k and v commute so that $v^{-1}wt^k$ act as the identity on an axis with terminal point P. Thus P is not in the image of π (we defined π on the complement of the set of singular points). $\square$

The set of singular points obviously has λ-measure zero, where λ is the Poisson measure. This is also true for the sets $\pi^{-1}(x)$ with $x \in V(\mathcal{T})$. Indeed this follows from Item (2) of Lemma 6.21 and the following

Lemma 6.22. *(see also* [49]*, Lemma 2.2.2) Let G be a countable group, μ a probability measure on G, and B a G-space endowed with a μ-stationary probability measure ν. Let $\pi : B \to C$ be any G-equivariant quotient map on a G-space C on which G acts with infinite orbits. Then all the fibers $\pi^{-1}(x)$ have ν-measure zero.*

Proof. Let $X = \pi^{-1}(x)$ be a fiber and denote by S the stabilizer of X in G. Consider the function f defined on G/S by $f(\gamma) = \nu(gX)$ where $\gamma = gS \in G/S$. One has $\sum_{\gamma\in G/S} f(\gamma) \leq \nu(B) < +\infty$ therefore f has a maximum value. On the other hand, since ν is μ-stationary, the function f satisfies the following mean-value property : $\sum_{h\in G} \mu(h) f(h^{-1}\gamma) = f(\gamma)$, therefore f must be constant. Since the G/S is infinite, f must be zero. $\square$

Thus, by Item (1) of Theorem 6.4 almost every sample path $\mathbf{x} = \{x_n\}$ converges to some $x_\infty \in \partial\widetilde{\mathcal{T}}$ for the measure associated to $\pi_*\lambda$. Observe however that there is again a slight abuse here since $\pi_*\lambda$ is a priori only defined on $\pi(\partial\widetilde{\mathcal{T}}^2 \setminus \{\text{singular points}\})$, which is only a subset of $\partial\widetilde{\mathcal{T}}$ when deprived of the zero measure set $V(\widetilde{\mathcal{T}})$, but it suffices to extend it by declaring the complement to be of measure zero. From which precedes $\pi_*\lambda$ is non-atomic and this is the hitting measure for the topology we have on $\mathcal{G} \cup \partial\widetilde{\mathcal{T}}$.

Observe that $\partial\widetilde{\mathcal{T}}$ satisfies the proximality property of Lemma 1.1. By Remark 1.2 and Lemma 1.1 the measure $\pi_*\lambda$ is thus the unique μ-stationary measure.

By Item (3) of Theorem 6.4 and Lemma 6.21, $(\partial\widetilde{\mathcal{T}}, \pi_*\lambda)$ is the Poisson boundary of $\mathcal{G}$. $\square$

6.1. **Consequence of Theorems 6.2 and 6.4.**

Theorem 6.23. *Let G be a torsion free hyperbolic group with infinitely many ends. Let $G_\Phi = G \rtimes_\Phi \mathbb{Z}$ be the semi-direct product of G with $\mathbb{Z}$ over a polynomially growing outer automorphism Φ.*

Let μ be a probability measure on G_Φ whose support generates G_Φ as a semi-group. Then there exist a simplicial Φ-invariant G-tree $\mathcal{T}$ and a bi-infinite geodesic A in $\mathcal{T}$ such that, if $\widetilde{\mathcal{T}}$ denotes the space obtained from $\widehat{\mathcal{T}}$ by closing each geodesic in $G.A$, then:

- *There is a topology on $G_\Phi \cup \partial\widetilde{\mathcal{T}}$ such that $\mathbf{P}$-almost every sample path $\mathbf{x} = \{x_n\}$ admits a subsequence which converges to some $x_\infty \in \partial\widetilde{\mathcal{T}}$.*
- *The hitting measure λ is non-atomic and it is the unique μ-stationary measure on $\partial\widetilde{\mathcal{T}}$.*
- *If μ has a finite first moment then the measured space $(\partial\widetilde{\mathcal{T}}, \lambda)$ is the Poisson boundary of (G_Φ, μ).*

Proof. In the case of a direct product of a free group with $\mathbb{Z}$, the construction of Section 3.1 gives a tree $\mathcal{T}$ as required by Theorem 6.2. This last theorem gives the conclusion in this case.

In the case of a cyclic extension over a polynomially growing automorphism, Theorem 4.5 gives a tree $\mathcal{T}$ which satisfies the properties required by Theorem 6.2, at the possible exception of the fact that α fixes each $\mathrm{Stab}_G(x)$-orbit of direction. However, the former theorem gives the finiteness of number of $\mathrm{Stab}_G(x)$-orbits of directions at each vertex x. Thus, after substituting Φ by Φ^k, so passing from G_Φ to the finite index subgroup G_{Φ^k}, we get a tree $\mathcal{T}$ for G_{Φ^k} as required by Theorem 6.2. This last theorem gives the conclusion for G_{Φ^k} and the conclusion for G_Φ follows from Theorem 1.5. $\square$

As in the proof of Corollary 5.11, the existence of the LL-map $\mathcal{Q}\colon \partial G \to \mathcal{T}$ gives:

Corollary 6.24. *With the assumptions and notations of Theorem 6.23:*

1. *There is a topology on $G_\Phi \cup \partial G$ such that $\mathbf{P}$-almost every sample path admits a subsequence which converges to some $x_\infty \in \partial G$.*
2. *The hitting measure λ on ∂G is non-atomic and this is the unique μ-stationary measure λ on ∂G.*
3. *If μ has a finite first moment, then the measured space $(\partial G, \lambda)$ is the Poisson boundary of $(\mathcal{G}, \mu)$.*

Remark 6.25. In the case of a direct product there is no need to pass to a finite-index subgroup so that the conditions on the measure may be relaxed to "finite first logarithmic moment and finite entropy" in Theorem 6.23 and Corollary 6.24. Moreover there is no need to pass to a subsequence in the first items of these results.

7. Extensions by non-cyclic groups

The plan of this section is parallel to the plan of the previous one. As a main goal we have the following

Theorem 7.1. *Let $\mathcal{G}$ be any one of the following two kinds of groups:*

- *A direct product $G \times \mathbb{F}_k$ where G is a torsion free hyperbolic group with infinitely many ends.*

- *A semi-direct product $G \rtimes_\theta \mathcal{P}$ of a free group $G = \mathbb{F}_n$ with a finitely generated subgroup $\mathcal{U}$ over a monomorphism $\theta\colon \mathcal{P} \to \mathrm{Out}(\mathbb{F}_n)$ such that $\theta(\mathcal{P})$ consists entirely of polynomially growing outer automorphisms.*

Let μ be a probability measure on $\mathcal{G}$ whose support generates $\mathcal{G}$ as a semi-group. Then there exists a simplicial $\theta(\mathcal{U})$-invariant G-tree $\mathcal{T}$, and a set $\mathcal{A}$ of bi-infinite geodesics in $\mathcal{T}$ whose union forms a proper subtree of $\mathcal{T}$ such that, if $\widetilde{\mathcal{T}}$ denotes the space obtained from $\widehat{\mathcal{T}}$ by closing each axis in $G.\mathcal{A}$, then:

(1) There is a topology on $\mathcal{G} \cup \partial\widetilde{\mathcal{T}}$ such that $\mathbf{P}$-almost every sample path admits a subsequence which converges to $x_\infty \in \partial\widetilde{\mathcal{T}}$.

(2) The hitting measure λ is non-atomic and this is the unique μ-stationary measure on $\partial\widetilde{\mathcal{T}}$.

(3) If μ has finite first moment, then the measured space $(\partial\widetilde{\mathcal{T}}, \lambda)$ is the Poisson boundary of $(\mathcal{G}, \mu)$.

As in the previous section, we will first focus on an intermediate result involving the product of two copies of a G-invariant tree.

Definition 7.2. Let $\theta_0, \theta_1 \colon \mathcal{U} \to \mathrm{Aut}(\mathbb{F}_n)$ be two monomorphisms with $[\theta_0(\mathcal{U})] = [\theta_1(\mathcal{U})] := \theta(\mathcal{U})$ in $\mathrm{Out}(G)$. Let $\mathcal{T}$ be a $\theta(\mathcal{U})$-invariant G-tree.

The (θ_0, θ_1)*-action of $\mathcal{G} = G \rtimes_\theta \mathcal{U}$ on $\mathcal{T} \times \mathcal{T}$* is the action given by $\Theta\colon \mathcal{G} \to \mathrm{Isom}(\mathcal{T} \times \mathcal{T})$ with:

$$\Theta(wu) \left\{ \begin{array}{ccccccc} \mathcal{T} & \times & \mathcal{T} & \to & \mathcal{T} & \times & \mathcal{T} \\ (P & , & Q) & \mapsto & (wH^{-1}_{\theta_0(u)}(P) & , & wH^{-1}_{\theta_1(u)}(Q)) \end{array} \right.$$

Theorem 7.3. *Let $\mathcal{G}$ be any one of the following two kinds of groups:*

- *A direct product $G \times \mathbb{F}_k$ where G is a torsion free hyperbolic group with infinitely many ends.*
- *A semi-direct product $G \rtimes_\theta \mathcal{P}$ of a free group $G = \mathbb{F}_n$ with a finitely generated subgroup $\mathcal{P}$ over a monomorphism $\theta\colon \mathcal{P} \to \mathrm{Out}(\mathbb{F}_n)$ such that $\theta(\mathcal{P})$ consists entirely of polynomially growing outer automorphisms.*

Let μ be a probability measure on $\mathcal{G}$ whose support generates $\mathcal{G}$ as a semi-group.
There exist

- *two monomorphisms $\theta_0, \theta_1 \colon \mathcal{P} \to \mathrm{Aut}(G)$ with $[\theta_0(\mathcal{P})] = [\theta_1(\mathcal{P})] = \theta(\mathcal{P})$ and $\mathcal{G} = G \rtimes_\theta \mathcal{P}$,*
- *a simplicial $\theta(\mathcal{U})$-invariant G-tree $\mathcal{T}$,*
- *a set $\mathcal{A}$ of bi-infinite geodesics in $\mathcal{T}$ whose union forms a proper subtree of $\mathcal{T}$*

such that, if $\widetilde{\mathcal{T}}$ denotes the space obtained from $\widehat{\mathcal{T}}$ by closing each geodesic in $G.\mathcal{A}$ and $\partial\overline{\mathcal{G}.O}$ denotes the boundary of the closure of the (θ_0, θ_1)-orbit of O in $\partial(\widetilde{\mathcal{T}} \times \widetilde{\mathcal{T}})$ then:

(1) $\mathbf{P}$-almost every sample path $\mathbf{x} = \{x_n\}$ admits a subsequence which converges to some $x_\infty \in \partial\overline{\mathcal{G}.O}$.

(2) The hitting measure λ is a non-atomic measure and this is the unique μ-stationary probability measure on $\partial\overline{\mathcal{G}.O}$.

(3) If the measure μ has finite first moment with respect to a word-metric on $\mathcal{G}$ then the measured space $(\partial\overline{\mathcal{G}.O}, \lambda)$ is the Poisson boundary of $(\mathcal{G}, \mu)$.

Proof. The proof follows exactly the same scheme as the proof of Theorem 6.4. The only difference lies in the singular elements.

Let us first consider the case where $\mathcal{G}$ is a direct product $G \times \mathbb{F}_2$ (there is no loss of generality in taking $\mathbb{F}_2$ instead of $\mathbb{F}_k$). We first choose a non trivial $\alpha \in \mathrm{Inn}(G)$ and construct a α-invariant G-tree $\mathcal{T}$ as in the previous section. Let t_1, t_2 be the two generators of $\mathbb{F}_2$ and let $x_1, \cdots, x_r$ be the generators of G. We make them act on $\mathcal{T}$ as two elliptic isometries fixing a unique vertex v_i, $v_1 \neq v_2$ and v_1, v_2 are the two vertices of some edge E: t_1 acts as α whereas t_2 acts as $x_1 \alpha x_1^{-1}$ for instance. This gives the θ_0-action. For each t_i there is a unique $g_i \in G$ such that $g_i t_i$ fixes E. Then we consider another copy of $\mathcal{T}$ and make act the t_i's in such a way that $g_i t_i$ acts as a hyperbolic isometry H_i and the subgroup $\mathcal{H} = \langle H_1, H_2 \rangle$ is free. This gives the θ_1-action. Now the subtree $T \subsetneq \mathcal{T}$ announced by Theorem 7.3 is the tree of this free subgroup. We consider the collection of bi-infinite geodesics $\mathcal{A}'$ given by the collection of the axis of the elements of $\mathcal{H}$ and the bi-infinite geodesics which are accumulations of such axis. The set $\{(X, Y) \in \partial T \times \partial T \text{ s.t. } \exists\, A \in \mathcal{A}' \text{ with } X, Y \in \partial A\}$ is a closed subset of $\partial \mathcal{T} \times \partial \mathcal{T}$. Thus closing the bi-infinite geodesics in $\mathcal{A}'$ yields, as in the previous section, a Hausdorff, compact space. We now set $\mathcal{A} = G\mathcal{A}'$ and we close the bi-infinite geodesics in $\mathcal{A}$. The space $\widetilde{\mathcal{T}}$ we get by closing the bi-infinite geodesics in $\mathcal{A}$ is Hausdorff and compact by the same argument as in the previous section. The (θ_0, θ_1)-action on $\mathcal{T} \times \mathcal{T}$ is the action made explicit above. All the arguments until the end are then copies of already given arguments.

The adaptation to the case of the extension of a free group by a polynomial growth subgroup is done as follows: Theorem 4.10 gives a finite-index subgroup $\mathcal{U}$ for which there exists a $\theta_{|\mathcal{U}}(\mathcal{U})$-invariant $\mathbb{F}_n$-tree $\mathcal{T}$. Assuming without loss of generality that $\mathcal{U}$ is 2-generated, there exist as before $g_i \in G$ such that $g_i t_i$ both fix the same edge E since the t_i's fix each $\mathrm{Stab}_G(v)$-orbit of directions at v. We leave the reader work out the details for the remaining arguments. We conclude at the end by Theorem 1.5 which allows us to recover the conclusion for the original group $G \rtimes_\theta \mathcal{P}$, of which $G \rtimes_{\theta_{|\mathcal{U}}} \mathcal{U}$ is a finite-index subgroup. □

Theorem 7.1 is deduced from Theorem 7.3 in the same way as Theorem 6.2 was deduced from Theorem 6.4: Lemma 6.21 still holds here. □

As was previously done, the existence of the LL-map $\mathcal{Q}$ gives the

Corollary 7.4. *With the assumptions and notations of Theorem 7.1:*

(1) There is a topology on $\mathcal{G} \cup \partial G$ such that $\mathbf{P}$-almost every sample path admits a subsequence which converges to $x_\infty \in \partial G$.

(2) The hitting measure λ is non-atomic and this is the unique μ-stationary measure on ∂G.

(3) If μ has a finite first moment, then the measured space $(\partial G, \lambda)$ is the Poisson boundary of $(\mathcal{G}, \mu)$.

Remark 7.5. In the case where $\mathcal{G}$ is a direct product or is a semi-direct product over a *unipotent subgroup* (see Theorem 4.10) of polynomially growing automorphisms then there is no need to pass to a finite-index subgroup so that the conditions on the measure in item (3) may be relaxed to "finite first logarithmic moment and finite entropy" in Theorem 7.1 and Corollary 7.4. Moreover there is no need to pass to a subsequence in the first items of these results.

8. Examples

8.1. **Extension of a free group by a free group of polynomially growing automorphisms.** Let $\mathbb{F}_3 = \langle a, b, c\rangle$ and let $\mathcal{U} := \mathbb{F}_2 = \langle t_1, t_2\rangle$. We define $\theta(t_1) := \alpha \in \mathrm{Aut}(\mathbb{F}_3)$ and $\theta(t_2) := \beta \in \mathrm{Aut}(\mathbb{F}_3)$ by:

(1) $\alpha(a) = a$, $\alpha(b) = b$ and $\alpha(c) = ca$.
(2) $\beta(a) = a$, $\beta(b) = b$ and $\beta(c) = cb$.

The morphism $\theta\colon \mathcal{U} \to \mathrm{Aut}(\mathbb{F}_n)$ is a monomorphism, i.e. $\langle \alpha, \beta\rangle$ is a free subgroup of rank 2 of $\mathrm{Aut}(\mathbb{F}_3)$ because, for any non-trivial reduced word u in $\alpha^{\pm 1}, \beta^{\pm 1}$, the word $u(c)$ is non-trivial. Hence there are no non-trivial relation between $\alpha^{\pm 1}$ and $\beta^{\pm 1}$.

The subgroup $\langle \alpha, \beta\rangle$ is a subgroup of polynomially growing automorphisms. Indeed a and b have 0-growth under α and β, so under any $u \in \mathcal{U}$, whereas c has linear growth. Hence for any $w \in \mathbb{F}_3$ the length of $u(w)$ growths at most linearly with respect to $|u|_{\langle a,b,c\rangle}$.

8.2. **Extension of a free group by an abelian group of polynomially growing automorphisms.** Let $\alpha \in \mathrm{Aut}(\mathbb{F}_2)$, $\mathbb{F}_2 = \langle a, b\rangle$, be defined by $\alpha(a) = a$, $\alpha(b) = ab$. This is a polynomially growing automorphism.

We consider the free group $\mathbb{F}_{2k} = \langle a_1, b_1, \cdots, a_k, b_k\rangle$ and the automorphisms $\alpha_1, \cdots, \alpha_k \in \mathrm{Aut}(\mathbb{F}_{2k})$ defined by

$$\alpha_i(a_i) = a_i \text{ for any } i = 1, \cdots, k$$
$$\alpha_i(b_i) = b_i a_i \text{ if } i = j$$
$$\alpha_i(b_j) = b_j \text{ if } i \neq j.$$

The aubgroup $\langle \alpha_1, \cdots, \alpha_k\rangle < \mathrm{Aut}(\mathbb{F}_{2k})$ is a $\mathbb{Z}^k$-subgroup of polynomially growing automorphisms in $\mathrm{Aut}(\mathbb{F}_{2k})$.

8.3. **The braid group B_3.** Let $\alpha \in \mathrm{Aut}(\mathbb{F}_2)$, $\mathbb{F}_2 = \langle x_1, x_2\rangle$, be defined by $\alpha(x_1) = x_1x_2x_1x_2^{-1}x_1^{-1}$, $\alpha(x_2) = x_1x_2x_1^{-1}$. Since x_1x_2 is fixed, one easily checks that α is a polynomially growing automorphism. The group $\mathbb{F}_2 \rtimes_\alpha \mathbb{Z}$ is isomorphic to P_3, the pure braid group over three strands, which is a subgroup of index 6 in the braid group B_3. Let us give some details. The braid group B_3 admits the Artin presentation $\langle a, b | aba = bab\rangle$ and P_3 is the subgroup generated by a^2, b^2, ba^2b^{-1}. The subgroup of P_3 generated by $x_1 = ba^2b^{-1}$ and $x_2 = b^2$ is free over $\{x_1, x_2\}$ and normal in P_3 with $P_3/\langle x_1, x_2\rangle$ being isomorphic to $\langle a^2\rangle$. The conjugacy by a^{-2} in $\langle x_1, x_2\rangle$ is given by α (see [13, 14]). Combining Theorem 1.5 and Theorem 6.23 we obtain:

Proposition 8.1. *Let μ be a probability measure on the braid group B_3 with a finite first moment. Then there exists a unique non-atomic stationary measure λ on $\partial\mathbb{F}_2$ and $(\partial\mathbb{F}_2, \lambda)$ is the Poisson boundary of (B_3, μ).*

See [27] for a description of the Poisson boundary of the braid group B_n in terms of projective measured foliations on the $n + 1$-punctured 2-sphere, and [58] for a detailed study of nearest neighbour random walks on B_3.

9. Relative hyperbolicity and application to Poisson boundaries

The aim of this section is to give another description of the Poisson boundary of a group $G_\alpha = G \rtimes_\alpha \mathbb{Z}$ when G is either a free group or the fundamental group of a compact hyperbolic surface, and α is an exponentially growing automorphism. There seems to be nothing really new here for geometric group theorists, but only a compilation of folklore, well-known and more recent results.

9.1. **Poisson boundary of strongly relatively hyperbolic groups.** The Proposition 9.1 below describes the Poisson boundary of a strongly relatively hyperbolic group in terms of its relative hyperbolic boundary. It can be easily deduced from [48]. Beware however that the hyperbolicity of the coned-off Cayley graph (see below) of Farb [26] is not sufficient: we really need the strong version of the relative hyperbolicity. We briefly recall some basic facts about relative hyperbolicity and otherwise invite the interested reader to consult [26], [15] for various definitions of relative hyperbolicity. If G is a discrete group with generating set S, and H a is a subgroup of G, the *coned-off Cayley graph* $\Gamma_S^H(G)$ *of* (G, H) is the graph obtained from by adding to $\Gamma_S(G)$ (the Cayley graph of G with respect to S) a vertex $v(gH)$ for each left H-class and an edge of length $\frac{1}{2}$ between $v(gH)$ and all the vertices of $\Gamma_S(G)$ associated to elements in the class gH. The *weak relative hyperbolicity of* G *with respect to* H just requires the hyperbolicity of $\Gamma_S^H(G)$. The *strong relative hyperbolicity* requires in addition that $\Gamma_S^H(G)$ satisfy the so-called *Bounded Coset Penetration* property, see [26]. We will not recall its definition here but just say that this property forbids that two left H-classes remain parallel along arbitrarily long paths in $\Gamma_S(G)$. Combined with the hyperbolicity of $\Gamma_S^H(G)$, it follows that any two left H-classes separate exponentially so that, in particular, they define distinct points in the *relative hyperbolic boundary* $\partial^{RH}(G, H)$. Of course some care has to be taken in order to get a correct definition of this boundary, since $\Gamma_S^H(G)$ is not a proper space (closed balls are not compact) as soon as H is infinite. We refer the reader to [15] or [70] to the definition of this relative hyperbolic boundary. The topology used by Bowditch is close in spirit to the observers topology used before in the current paper to deal with the non-properness of $\mathbb{R}$-trees. Another construction of this relative hyperbolic boundary can be found in [42] where the author takes more care in constructing a proper space associated to (G, H) but then gets the relative hyperbolic boundary in an easier way. This last definition is closer in spirit to Gromov approach of relative hyperbolicity as initiated a long time ago in the seminal paper [41]. Of course all these notions have a straightforward generalization when substituting a finite family of subgroups $\mathcal{H}$ to a single subgroup H.

Proposition 9.1. *Let G be a finitely generated group which is strongly hyperbolic relatively to a finite family of subgroups $\mathcal{H}$. Let $\partial^{RH}(G, \mathcal{H})$ be the relative hyperbolic boundary of $(G, \mathcal{H})$. Let μ be a probability measure on G whose support generates G as a semi-group. Then:*

(1) $\mathbf{P}$*-almost every sample path $\mathbf{x} = \{x_n\}$ converges to some $x_\infty \in \partial^{RH}(G, \mathcal{H})$.*
(2) The measure $\lambda = \pi(\mathbf{P})$ on $\partial^{RH}(G, \mathcal{H})$ is non-atomic and is such that $(\partial^{RH}(G, \mathcal{H}), \lambda)$ is a μ-boundary of (G, μ). It is the unique μ-stationary measure on $\partial^{RH}(G, \mathcal{H})$.
(3) If μ has finite first logarithmic moment and finite entropy then the measured space $(\partial^{RH}(G, \mathcal{H}), \lambda)$ is the Poisson boundary of (G, μ).

9.2. **Applications to cyclic extensions of free and surface groups.** We need now some material borrowed from [37]. Let $\Phi \in \mathrm{Out}(\mathbb{F}_n)$. A family of Φ-polynomially growing subgroups $\mathcal{H} = (H_1, \cdots, H_r)$ is called *exhaustive* if every element $g \in \mathbb{F}_n$ of polynomial growth is conjugate to an element contained in some of the H_i. The family $\mathcal{H}$ is called *minimal* if no H_i is a subgroup of any conjugate of some H_j with $i \neq j$. The following proposition is well-known among the experts of free group automorphisms, see [53] or [37] for a proof.

Proposition 9.2. [53, 37] *Every outer automorphism $\Phi \in \mathrm{Out}(\mathbb{F}_n)$ possesses a Φ-*characteristic family $\mathcal{H}(\Phi)$*, that is a family satisfying the following properties:*

(a) $\mathcal{H}(\Phi) = (H_1, \ldots, H_r)$ *is a finite, exhaustive, minimal family of finitely generated subgroups* H_i *that are of polynomial growth.*

(b) The family $\mathcal{H}(\Phi)$ *is uniquely determined, up to permuting the* H_i *or replacing any* H_i *by a conjugate.*

(c) The family $\mathcal{H}(\Phi)$ *is* Φ*-invariant (up to conjugacy).*

We need to precise a little bit more this notion of "invariance" for a family of subgroups, with respect to the action of an automorphism (and not only an outer automorphism).

For any $\alpha \in \mathrm{Aut}(\mathbb{F}_n)$, a family of subgroups $\mathcal{H} = (H_1, \ldots, H_r)$ is called *α-invariant up to conjugation* if there is a permutation σ of $\{1, \ldots, r\}$ as well as elements $h_1, \ldots, h_r \in G$ such that $\alpha(H_k) = h_k H_{\sigma(k)} h_k^{-1}$ for each $k \in \{1, \ldots, r\}$. Let $\mathcal{H} = (H_1, \ldots, H_r)$ be a finite family of subgroups of G which is α-invariant up to conjugacy. For each H_i in $\mathcal{H}$ let $m_i \geq 1$ be the smallest integer such that $\alpha^{m_i}(H_i)$ is conjugate in G to H_i, and let h_i be the conjugator: $\alpha^{m_i}(H_i) = h_i H_i h_i^{-1}$. We define the *induced mapping torus subgroup*:

$$H_i^\alpha = \langle H_i, h_i^{-1} t^{m_i} \rangle \subset G_\alpha$$

Definition 9.3. Let $\mathcal{H} = (H_1, \ldots, H_r)$ be a finite family of subgroups of G which is α-invariant up to conjugacy. A family of induced mapping torus subgroups

$$\mathcal{H}^\alpha = (H_1^\alpha, \ldots, H_q^\alpha)$$

as above is the *mapping torus of $\mathcal{H}$ with respect to α* if it contains for each conjugacy class in G_α of any H_i^α, for $i = 1, \ldots, r$, precisely one representative.

The following theorem is from [37] in the case where $G = \mathbb{F}_n$ and from [36] (see also [39]) in the surface case.

Theorem 9.4. *The group* $G \rtimes_\Phi \mathbb{Z}$ *is strongly hyperbolic relative to the mapping-torus of a* Φ*-characteristic family.*

We can now state the theorem of this section, which follows directly from Proposition 9.1 and Theorem 9.4:

Theorem 9.5. *Let* G *be the fundamental group of a compact hyperbolic surface or the free group of rank* n*. Let* $\Phi \in \mathrm{Out}(G)$ *be an exponentially growing outer automorphism and let* α *be any automorphism in the class* Φ*. Let* μ *be a probability measure on* $G_\alpha = G \rtimes_\alpha \mathbb{Z}$ *whose support generates* G_α *as a semi-group. Then:*

(1) $\mathbf{P}$*-almost every sample path* $\mathbf{x} = \{x_n\}$ *converges to some* $x_\infty \in \partial^{RH}(G_\alpha, \mathcal{H}^\alpha(\Phi))$.

(2) The hitting measure λ*, which is the distribution of* x_∞*, is a non-atomic measure on* $\partial^{RH}(G_\alpha, \mathcal{H}^\alpha(\Phi))$ *such that* $(\partial^{RH}(G_\alpha, \mathcal{H}^\alpha(\Phi)), \lambda)$ *is a* μ*-boundary of* (G_α, μ) *and* λ *is the unique* μ*-stationary probability measure on* $\partial^{RH}(G_\alpha, \mathcal{H}^\alpha(\Phi))$.

(3) If the measure μ *has finite first logarithmic moment and finite entropy with respect to a word-metric on* G_α*, then the measured space* $(\partial^{RH}(G_\alpha, \mathcal{H}^\alpha(\Phi)), \lambda)$ *is the Poisson boundary of* (G_α, μ).

When the outer automorphism is hyperbolic, meaning that any conjugacy-class has exponential growth under Φ, the Φ-characteristic family is trivial. This is exactly the case where G_α is a hyperbolic group [9, 38, 36] so that, in this case, Theorem 9.5 tells nothing new with respect to [48].

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