GEODESICS IN TREES OF HYPERBOLIC AND RELATIVELY
HYPERBOLIC GROUPS

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Abstract. We present a careful approximation of the quasi geodesics in trees of hyperbolic or relatively hyperbolic groups. As an application we prove a combination theorem for finite graphs of relatively hyperbolic groups, with both Farb’s and Gromov’s definitions.

1. Introduction

The main part of this paper is devoted to give a precise description of the (quasi) geodesics in trees of hyperbolic and relatively hyperbolic groups. Such a work might appear not very appealing, and somehow quite technic. In order to show that this however might be worthy, let us give an application: a combination theorem for hyperbolic and relatively hyperbolic groups. That is, a theorem giving a condition for the fundamental group of a graph of relatively hyperbolic groups being a relatively hyperbolic group. In [3] (see also [23]), the authors introduce the notion of (finite) qi-embedded graph of groups and spaces $\mathcal{G}$. Then, assuming the Gromov hyperbolicity of the vertex spaces and the quasiconvexity of the edge spaces in the vertex spaces, they give a criterion for the hyperbolicity of the fundamental group of $\mathcal{G}$. Since then different proofs have appeared, which treat the so-called ‘acylindrical case’: see, among others, [20, 25]. Acylindrical means that the fixed set of the action of any element of the fundamental group of the graph of groups on the universal covering has uniformly bounded diameter. The non-acylindrical case is less common: see [24] which relies on [3] but clarifies its consequences when dealing with a certain class of mapping-tori of injective, non surjective free group endomorphisms, or [13] which, by an approach similar to the one presented here, gives a new proof of [3] in the case of mapping-tori of free group endomorphisms. Nowadays the attention has drifted from hyperbolic groups to relatively hyperbolic groups. A notion of relative hyperbolicity was already defined by Gromov in his seminal paper [21]. Since then it has been revisited and elaborated on in many papers. Two distinct definitions now coexist. In parallel to the Gromov relative hyperbolicity, sometimes called strong relative hyperbolicity, there is the sometimes called weak relative hyperbolicity introduced by Farb [11]. Bowditch [5] and Osin [27] give alternative definitions, but which are equivalent either to Farb’s or to Gromov’s definition. In fact, it has been proved [8, 27] (see also [5]) that Gromov definition is equivalent to Farb definition plus an additional property termed Bounden

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Coset Penetration property (BCP in short), due to Farb [11]. Relatively hyperbolic groups in the strong (that is Gromov) sense form a class encompassing hyperbolic groups, fundamental groups of geometrically finite orbifolds with pinched negative curvature, groups acting on CAT(0)-spaces with isolated flats among many others. First combination theorems in some particular (essentially acylindrical) cases have been given in the setting of the relative hyperbolicity: [1], [10] or [28, 29]. One result [16] treats a particular non-acylindrical case, namely the relative hyperbolicity of one-ended hyperbolic by cyclic groups. Since a first version of this paper was written, a paper [26] has appeared, giving a combination theorem dealing with more general non-acylindrical cases than [16]: the authors heavily rely upon [3], which they use as a “black-box”. Getting such a “general” combination theorem for relatively hyperbolic groups is one of the questions (attributed to Swarup) raised in Bestvina’s list [2]. We offer here an answer, as an application of our work on geodesics in trees of spaces. We would like to emphasize at once that we do not appeal to [3], but instead give a new proof of it as a particular case. Where the authors of [3] use “second-order” geometric characterization of hyperbolicity via isoperimetric inequalities, we use “first-order” geometric characterization, via approximations of geodesics and the thin triangle property. At the expense of heavier and sometimes tedious computations, this naïve approach allows us to engulf in a same setting (at least when dealing with combination theorems) both absolute and relative hyperbolicity.

We are now going to give some group-theoretic results that will be deduced from the - perhaps more technical - theorems about trees of spaces that we prove further in the paper. Other aside results about groups are presented in Section 6.1. Whereas the paper essentially deals with metric spaces, we gather here in the introduction most of the definitions about groups (for the definition of strongly relatively hyperbolic groups however, see Definition 6.2). In this way the statements below will be - almost - self-contained.

We begin by a result about free extensions of relatively hyperbolic groups. It is easier to grasp than the more general combination theorems we give further. The relatively hyperbolic automorphisms we define below first appeared in [14] where we announced a (weak) version of the results of the present paper. They generalize the Gromov hyperbolic automorphisms [3].

**Definition 1.1.** Let $G = \langle S \rangle$ be a finitely generated group and let $\mathcal{H} = \{H_1, \cdots, H_k\}$ be a finite family of subgroups of $G$.

The $\mathcal{H}$-word metric $|.|_{\mathcal{H}}$ is the word-metric for $G$ equipped with the (usually infinite) set of generators which is the union of $S$ with the elements of $G$ in the subgroups of the collection $\mathcal{H}$.

Let $\text{Aut}(G)$ be the group of all automorphisms of $G$. Let $\alpha \in \text{Aut}(G)$. We say that $\alpha$ is a relative automorphism of $(G, \mathcal{H})$ if and only if there is a permutation $\sigma$ of $\{1, \cdots, k\}$ such that for any $H_i \in \mathcal{H}$ there is $g_i \in G$ with $\alpha(H_i) = g_i^{-1}H_{\sigma(i)}g_i$. If $\sigma$ is the identity, we say that $\alpha$ fixes $\mathcal{H}$ or fixes each $H_i$ up to conjugacy. We denote by $\text{Aut}(G, \mathcal{H}) < \text{Aut}(G)$ the subgroup of all relative automorphisms of $(G, \mathcal{H})$.

Let $\alpha \in \text{Aut}(G)$. We say that $\alpha$ is hyperbolic relative to $\mathcal{H}$ if and only if $\alpha \in \text{Aut}(G, \mathcal{H})$ and there exist $\lambda > 1$ and $M, N \geq 1$ such that for any $w \in G$ with $|w|_{\mathcal{H}} \geq M$:

$$\lambda|w|_{\mathcal{H}} \leq \max(|\alpha^N(w)|_{\mathcal{H}}, |\alpha^{-N}(w)|_{\mathcal{H}}).$$

This definition is slightly more general than the definition given in [14]. The constant $M$ did not appear there. It is however more natural: thanks to this additional constant $M$, the definition is obviously invariant under conjugacy.$^1$

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$^1$The author is grateful to F. Dahmani, V. Guirardel and M. Lustig for this observation.
**Definition 1.2.** Let $G$ be a finitely generated group and let $\mathcal{H}$ be a finite family of subgroups of $G$.

A pair $(\mathfrak{A}, \iota)$ where $\mathfrak{A}$ is a finitely generated group and $\iota: \mathfrak{A} \hookrightarrow \text{Aut}(G, \mathcal{H})$ is a monomorphism defines a uniform group of relatively hyperbolic automorphisms of $(G, \mathcal{H})$ \(^2\) if and only if for any finite generating set $\mathcal{A}$ of $\mathfrak{A}$ there exist $\lambda > 1$ and $M, N \geq 1$ such that for any element $w \in G$ with $|w|_\mathcal{H} \geq M$, for any pair of elements $a_1, a_2 \in \mathfrak{A}$ with $|a_1|_\mathcal{A} = |a_2|_\mathcal{A} = N$ and $d(A(a_1, a_2) = 2N$, the following property is satisfied:

$$\lambda |w|_\mathcal{H} \leq \max(|\iota(a_1)(w)|_{\mathcal{H}}, |\iota(a_2)(w)|_{\mathcal{H}}).$$

In Definition 1.2 the existence of the constants $\lambda, M, N$ holds for any finite generating set $\mathcal{A}$ if and only if it holds for some such generating set. But $\lambda, M, N$ depend on the choice of $\mathcal{A}$.

**Definition 1.3.** Let $G$ be a finitely generated group and let $\mathcal{H} = \{H_1, \cdots, H_k\}$ be a finite family of subgroups of $G$. Let $r$ be a positive integer, let $\mathbb{F}_r$ be the free group of rank $r$ and let $\iota: \mathbb{F}_r \hookrightarrow \text{Aut}(G, \mathcal{H})$ be a monomorphism.

A $(\mathbb{F}_r, \iota)$-extension $\mathcal{H}_r$ of $\mathcal{H}$ is a maximal family of subgroups of $G \rtimes \mathbb{F}_r$ of the form $\langle H_i, a_{i,1}^{-1}g_{i,1}, \cdots, a_{i,m}^{-1}g_{i,m}, \cdots \rangle$ such that:

- Each $a_{i,m} \in \mathbb{F}_r$ satisfies $\iota(a_{i,m})(H_i) = g_{i,m}^{-1}H_ig_{i,m}$ and $\langle a_{i,1}, a_{i,2}, \cdots \rangle$ generates the subgroup of all the elements of $\mathbb{F}_r$ whose images under $\iota$ fix each $H_i$ up to conjugacy.
- If $\mathcal{H}_j, \mathcal{H}_j'$ are two distinct subgroups in $\mathcal{H}_r$, then $H_i \in \mathcal{H}_j \cap G, H_k \in \mathcal{H}_j' \cap G$ then no element of $\mathbb{F}_r$ conjugates $H_i$ to $H_k$ in $G \rtimes \mathbb{F}_r$.

**Remark 1.4.** The free subgroup of $\mathbb{F}_r$ whose image under $\iota: \mathbb{F}_r \hookrightarrow \text{Aut}(G, \mathcal{H})$ fix each subgroup $H_i \in \mathcal{H}$ up to conjugacy is finitely generated, see [15].

When $r = 1$ in the above definition, i.e. $\mathbb{F}_r = \langle t \rangle$, we get the easier notion of the mapping-torus of $\mathcal{H}$ under $\alpha \in \text{Aut}(G, \mathcal{H})$ with $\alpha = \iota(t)$. This is a maximal family $\mathcal{H}_\alpha$ of subgroups $\mathcal{H}_j \subset G_\alpha$ satisfying the following properties:

- $\mathcal{H}_j = \langle H_j, t^{n_j}g_j^{-1} \rangle$, where $n_j$ is the minimal integer such that there is $g_j \in G$ with $\alpha^{n_j}(H_j) = g_j^{-1}H_jg_j$;
- whenever $\mathcal{H}_j = \langle H_j, t^{n_j}g_j^{-1} \rangle$, $\mathcal{H}_k = \langle H_k, t^{n_k}g_k^{-1} \rangle$ are two distinct subgroups in $\mathcal{H}_\alpha$, no power of $t$ conjugates $H_j$ to $H_k$ in $G_\alpha$.

**Theorem 1.5.** Let $G$ be a finitely generated group and let $\mathcal{H}$ be a finite family of subgroups of $G$. Let $(\mathbb{F}_r, \iota)$ define a uniform free group of relatively hyperbolic automorphisms of $(G, \mathcal{H})$. If $G$ is weakly hyperbolic relative to $\mathcal{H}$, then $G \rtimes \mathbb{F}_r$ is weakly hyperbolic to $\mathcal{H}$. If $G$ is strongly hyperbolic relative to $\mathcal{H}$, then $G \rtimes \mathbb{F}_r$ is strongly hyperbolic relative to a $(\mathbb{F}_r, \iota)$-extension of $\mathcal{H}$.

When $r = 1$ in the above theorem, that is when the considered free group is just $\mathbb{Z}$, we get the classical “mapping-torus” case, that is the case of semi-direct products $G \rtimes \mathbb{Z}$ with $G$ a relatively hyperbolic group. Corollary 6.17 gives a concrete application, when $G$ is the fundamental group of a compact surface and $\mathbb{Z}$ acts on $G$ by an automorphism induced by a homeomorphism $h$ of the surface $S$. In this case the mapping-torus group $G \rtimes \mathbb{Z}$ is weakly hyperbolic relative to the family formed by the cyclic subgroups generated by the boundary loops of $S$, the subgroups defined (up to conjugacy) by the maximal subsurfaces

\(^2\)The author would like to thank M. Heusener for inciting him to correct a previous formulation of this definition, which was unnecessarily more restrictive.
of $S$ preserved up to isotopy by a power of the homeomorphism and with no pseudo-Anosov component, and the cyclic subgroups generated by the reduction curves which are not already contained in the previous subgroups. It is strongly hyperbolic relatively to the family of subgroups composed of the subgroups associated to the boundary tori (assume for simplicity that $S$ is orientable and that $h$ preserves the orientation), the subgroups associated to the 3-dimensional submanifolds which are the mapping-tori of the maximal non pseudo-Anosov components and the subgroups associated to the 2-dimensional tori which are the mapping-tori of the remaining reduction curves.

We now recall the definition of weak relative hyperbolicity. The approach is from [11]. If $S$ is a discrete set, the cone with base $S$ is the space $S \times [0, \frac{1}{2}]$ with $S \times \{0\}$ collapsed to a point: this is the vertex of the cone. This cone is considered as a metric space, with distance function $d_S((x, t), (y, t')) = t + t'$. Let $(X, d)$ be a geodesic (or quasi geodesic - see Section 2) space. Putting a cone over a discrete subset $S$ of $X$ consists of pasting to $X$ a cone with base $S$ by identifying $S \times \{1/2\}$ with $S \subset X$. The resulting metric space, called the $S$-coned space or more simply the coned-space, $(\hat{X}_S, d_S)$ is such that all the points in $S$ are now at distance $\frac{1}{3}$ from the vertex of the cone, termed exceptional vertex. The intervals from the points in $S$ to the exceptional vertex are the exceptional edges. The metric of the coned space is the coned or relative metric. If $S$ is a family of subsets of $(X, d)$, then the $S$-coned space $\hat{X}_S$ is the space obtained by putting a cone over each set in $S$.

**Definition 1.6.** [11] A metric space $(X, d)$ is weakly hyperbolic relative to a family of subsets $S$ if the $S$-coned space $(\hat{X}_S, d_S)$ is Gromov hyperbolic.

Let $G$ be a group with finite generating set $S$ and Cayley graph $\Gamma_S(G)$. Let $\mathcal{H} = \{H_1, \cdots\}$ be a family of infinite subgroups $H_i$ of $G$.

The group $G$ is weakly hyperbolic relative to $\mathcal{H}$ if $\Gamma_S(G)$ is weakly hyperbolic relative to the family composed of the left-classes $xH_i$. The associated coned space is denoted by $\Gamma_S^\mathcal{H}(G)$. The exceptional vertex over the left $H_i$-class of $g$ is denoted by $\nu(gH_i)$.

The subgroups $H_i$ in the family $\mathcal{H}$ are the parabolic subgroups of $G$.

The above definition is equivalent to require that $G$ equipped with the $\mathcal{H}$-word metric be hyperbolic. However the introduction of the cones and of the coned Cayley graphs above is needed to introduce farther in the paper the Bounded Coset Penetration property.

Since the ultimate goal is a theorem about graphs of relatively hyperbolic groups, we introduce some notations for graphs and graphs of groups. If $\Gamma$ is a graph, $V(\Gamma)$ (resp. $E(\Gamma)$) denotes its set of vertices (resp. oriented edges). For $e \in E(\Gamma)$ we denote by $e^{-1}$ the same edge with opposite orientation. If $p$ is an edge-path in $\Gamma$, in particular if $p$ is an edge, $i(p)$ (resp. $t(p)$) denotes the initial (resp. terminal) vertex of $p$. In a tree, given any two vertices $x, y$, we denote by $[x, y]$ the unique reduced edge-path from $x$ to $y$.

If $\Gamma$ is a graph equipped with a metric, we denote by $d_\Gamma$ the associated distance. Let us now consider a graph of groups $\mathcal{G}$. The group associated to the edge $e$ (an edge-group) is denoted by $G_e$, and $G_e = G_{e^{-1}}$. The group associated to the vertex $v$ (a vertex-group) is denoted by $G_v$. If $e$ is an oriented edge, there is a morphism $t_e: G_e \to G_{t(e)}$ associated to the graph of groups structure. If $p = e^{\epsilon_1}_{\epsilon_1} \cdots e^{\epsilon_k}_{\epsilon_k}$ is an edge-path in $\mathcal{G}$ with $i(p) = v$ and $t(p) = w$ then we denote by $t_p: G_v \to G_w$ the morphism, if it exists, with maximal domain in $G_v$ given by $t_{\epsilon_1} \circ \cdots \circ t_{\epsilon_1} \circ t_{\epsilon_1}^{-1}$.

**Definition 1.7.** A graph of weakly relatively hyperbolic groups (resp. of strongly relatively hyperbolic groups), denoted by $(\mathcal{G}, \mathcal{H}_{t_e}, \mathcal{H}_{e^{-1}})$, is a finite graph of finitely generated groups $\mathcal{G}$ satisfying the following properties:
(a) Each edge-group \( G_e \) and each vertex-group \( G_v \) is weakly hyperbolic (resp. strongly hyperbolic) relative to a specified, possibly empty, finite family of infinite subgroups \( \mathcal{H}_e \) and \( \mathcal{H}_v \).

(b) There are \( a \geq 1, b \geq 0 \) such that for any oriented edge \( e \), \( \iota_e \) is a \((a, b)\)-quasi isometric embedding from \((G_e, |.|_{\mathcal{H}_e})\) to \((G_{t(e)}, |.|_{\mathcal{H}_{t(e)}})\).

(c) For each oriented edge \( e \) and for each subgroup \( H \) in \( \mathcal{H}_e \), \( \iota_e(H) \) is conjugate to a subgroup of some \( H' \) in \( \mathcal{H}_{t(e)} \).

For the general definition of the tree of spaces appearing below, see Definition 2. We think that this is not essential for the understanding of the statements of the two main results (Theorems 1.9 and 1.12).

**Definition 1.8.** Let \( \mathcal{G} \) be a graph of groups. A \( \mathcal{G}\text{-tree of spaces} \) is a tree of spaces \((\hat{X}, T)\) satisfying the following properties:

- The tree \( T \) is the universal covering of the graph of groups \( \mathcal{G} \). The fundamental group of \( \mathcal{G} \) acts cocompactly, isometrically and properly discontinuously on \( \hat{X} \).
- Each edge-space \( X_e \) (resp. each vertex-space \( X_v \)) is a metric cellular complex equipped with a cocompact, isometric, properly discontinuous action of \( G_e \) (resp. of \( G_v \)). There is an identification of a subset of the 0-cells of \( X_e \) (resp. of \( X_v \)) with \( G_e \) (resp. with \( G_v \)).
- Each quasi isometric embedding \( j_e : X_e \to X_{t(e)} \) is a continuous map which induces a morphism from \( \pi_1(X_e/G_e) \) to \( \pi_1(X_{t(e)}/G_{t(e)}) \) which is conjugate to the morphism \( \iota_e : G_e \to G_{t(e)} \).

Let \((\mathcal{G}, \mathcal{H}_v, \mathcal{H}_e)\) be a graph of relatively hyperbolic groups. A \( \mathcal{G}\text{-relative tree of spaces} \) is a tree of spaces \((\hat{X}, T)\) such that there is a \( \mathcal{G}\text{-tree of spaces} \((\hat{X}, T)\) with respect to which the following properties are satisfied:

- Each edge-space \( \hat{X}_e \) (resp. vertex-space \( \hat{X}_v \)) is the \( \mathcal{H}_e \)-coned space (resp. \( \mathcal{H}_v \)-coned space) of \( X_e \) (resp. of \( X_v \)). The fundamental group of \( \mathcal{G} \) acts by homeomorphisms on \( \hat{X} \) and this action extends its isometric action on \( \hat{X} \).
- Each quasi isometric embedding \( j_e : \hat{X}_e \to \hat{X}_{t(e)} \) is a continuous map such that:
  - Its restriction to \( X_e \subset X_{t(e)} \) is equal to \( j_e : X_e \to X_{t(e)} \) (the quasi isometric embedding associated to the oriented edge \( e \) in the tree of spaces-structure of \( \hat{X} \)).
  - For any exceptional vertex \( v(gH_i) \) of \( \hat{X}_e \), \( \hat{j}_e(v(gH_i)) \) is, if it exists, an exceptional vertex \( v(g'H_j) \) lying over a set containing \( j_e(gH_i) \). The images under \( \hat{j} \) of the exceptional edges ending at \( v(gH_i) \) are left-translates of an edge-path in the 1-skeleton of \( \hat{X}_{t(e)} \) which is the concatenation of an edge-path in the 1-skeleton of \( \hat{X}_{t(e)} \) corresponding to a conjugacy element between \( \iota_{t(e)}(H_i) \) and a subgroup of \( H_j \), followed by an exceptional edge ending at \( v(g'H_j) \).

The notation \( j_e(gH_i) \) in the last item of the above definition makes sense because a (quasi dense) subset of the vertices of the edge- and vertex-spaces has been identified to the elements of the edge- and vertex-groups. In fact, \( j_e \) is defined over a left-translate of the domain of the morphism \( \iota_e \) by an element not in the domain of \( \iota_e \). It realizes a left-translate of this morphism by an element not in the image of \( \iota_e \).

We now state the combination theorem for graphs of weakly relatively hyperbolic groups. If \((\hat{X}, T)\) (resp. \((\hat{X}, T)\)) is a tree of spaces (resp. relative tree of spaces) and \( p \) is
an edge-path in $T$, then $f_p : \hat{X}_{i(p)} \to \hat{X}_{t(p)}$ (resp. $f_p : \hat{X}_{i(p)} \to \hat{X}_{t(p)}$) is defined as was defined $t_p$ before in the graph of groups setting. We recall that the edge- and vertex-spaces of $\hat{X}$ are equipped with the relative $\mathcal{H}_e$- and $\mathcal{H}_v$-metrics of the edge- and vertex-groups.

**Theorem 1.9.** Let $(G, \mathcal{H}_v, \mathcal{H}_e)$ be a graph of weakly relatively hyperbolic groups. If some relative $(G, \mathcal{H}_v, \mathcal{H}_e)$-tree of spaces $(\hat{X}, T)$ satisfies the following property:

There exist $\lambda > 1$ and integers $M, N \geq 1$ such that for any $v \in V(T)$, for any $\alpha, \beta \in V(T)$ with $d_T(\alpha, v) = d_T(\beta, v) = N$ and $d_T(\alpha, \beta) = 2N$, if $g, h, k, l \in G_v$ are such that $\hat{j}_{[v, \alpha]}(g), \hat{j}_{[v, \alpha]}(h), \hat{j}_{[v, \beta]}(k)$ and $\hat{j}_{[v, \beta]}(l)$ are well-defined and satisfy

$$d_{\hat{X}_v}(\hat{j}_{[v, \alpha]}(g), \hat{j}_{[v, \alpha]}(h)) < \lambda d_{\hat{X}_v}(g, h)$$

and

$$d_{\hat{X}_v}(\hat{j}_{[v, \beta]}(k), \hat{j}_{[v, \beta]}(l)) < \lambda d_{\hat{X}_v}(k, l)$$

then the projections of $g, h$ on a geodesic of $\hat{X}_v$ between $k$ and $l$ are at distance smaller than $M$ one from the other.

Then the fundamental group of $G$ is weakly hyperbolic relative to the family composed of all the parabolic subgroups of the vertex-groups.

Figure 1 illustrates, schematically, the property of the above theorem.

![Figure 1](image)

Figure 1

Observe that by setting $k = g$ and $l = h$ in the above theorem, the given property implies that for any $g, h \in G_v$ such that $\hat{j}_{[v, \alpha]}(g), \hat{j}_{[v, \alpha]}(h), \hat{j}_{[v, \beta]}(k)$ and $\hat{j}_{[v, \beta]}(l)$ are well-defined and such that $d_{\hat{X}_v}(g, h) \geq M$ we have

$$\max(d_{\hat{X}_v}(\hat{j}_{[v, \alpha]}(g), \hat{j}_{[v, \alpha]}(h)), d_{\hat{X}_v}(\hat{j}_{[v, \beta]}(g), \hat{j}_{[v, \beta]}(h))) \geq \lambda d_{\hat{X}_v}(g, h).$$

We pass to the strong relative hyperbolicity. The combination theorem we get in this case only deals with a restricted class of graphs of strongly relatively hyperbolic groups. This is because the description of the subgroups to put in the relative part is heavier in this setting then in the setting of weak relative hyperbolicity. We hope that the restriction we put is a not too bad compromise between clarity and generality.

**Definition 1.10.** A graph of strongly relatively hyperbolic groups $(G, \mathcal{H}_v, \mathcal{H}_e)$ is fine if there is $M \geq 0$ such that for each oriented edge $e$ of $G$, for each parabolic subgroup $H$ in $\mathcal{H}_e$

- either $t_e(H)$ is conjugate in $G_{t(e)}$ to a parabolic subgroup $H'$ in $\mathcal{H}_{t(e)}$,
- or for any $g \in G_{t(e)}$, for any edge $e'$ with $t(e') = t(e)$, for any parabolic subgroup $H'$ in $\mathcal{H}_{e'}$, the $\mathcal{H}_{t(e)}$-relative length of $g^{-1}t_e(H)g \cap t_{e'}(H')$ is bounded above by $M$.
Definition 1.11. Let \((G, H_v, H_e)\) be a fine graph of strongly relatively hyperbolic groups.

Two parabolic subgroups \(H \in H_v\) and \(H' \in H_w\) belong to a same parabolic orbit if there is an edge-path \(p\) in \(G\) with \(i(p) = v\) and \(t(p) = w\) such that the morphism \(i_p\) is well-defined over \(H\) and there exists \(g \in G_w\) with \(i_p(H) = g^{-1}H'g\).

A subgroup \(H \in H_v\) belongs to an infinite parabolic orbit if there is loop \(p\) through \(v\) and \(g \in G_v\) such that \(i_p(H) = g^{-1}Hg\).

The characteristic subgroup \(\mathfrak{F}\) of the fundamental group of \(G\) is the maximal free subgroup generated by the edges in the complement of a maximal tree in \(G\) such that:

- For each element \(p\) in \(\mathfrak{F}\), for each parabolic subgroup \(H\) in a vertex-group, either \(i_p\) fixes \(H\) up to conjugacy or \(i_p\) is not defined over \(H\).
- For each infinite parabolic orbit, there is a parabolic subgroup \(H\) in the orbit and an element \(p\) in \(\mathfrak{F}\) which fixes \(H\) up to conjugacy.

Let \(H \in H_v\) be a parabolic subgroup in an infinite orbit. A free extension of \(H\) is a subgroup of the fundamental group of \(G\) generated by \(H\) and by all the elements \(pg_p\) such that \(p\) belongs to the characteristic subgroup \(\mathfrak{F}\) and \(i_p(H) = g_p^{-1}Hg_p\).

Theorem 1.12. Let \((G, H_v, H_e)\) be a fine graph of strongly relatively hyperbolic groups. If some relative \((G, H_v, H_e)\)-tree of spaces \((\tilde{X}, T)\) satisfies the following properties:

(a) There exist \(\lambda > 1\) and integers \(M, N \geq 1\) such that for any \(v \in V(T)\), for any \(\alpha, \beta \in V(T)\) with \(d_T(\alpha, v) = d_T(\beta, v) = N\) and \(d_T(\alpha, \beta) = 2N\), if \(g, h, k, l \in G_v\) are such that \(\tilde{\jmath}_{[v, \alpha]}(g), \tilde{\jmath}_{[v, \alpha]}(h), \tilde{\jmath}_{[v, \beta]}(k)\) and \(\tilde{\jmath}_{[v, \beta]}(l)\) are well-defined and satisfy
\[
d_{\mathfrak{F}_v}(\tilde{\jmath}_{[v, \alpha]}(g), \tilde{\jmath}_{[v, \alpha]}(h)) < \lambda d_{\mathfrak{F}_v}(g, h)
\]
and
\[
d_{\mathfrak{F}_v}(\tilde{\jmath}_{[v, \beta]}(k), \tilde{\jmath}_{[v, \beta]}(l)) < \lambda d_{\mathfrak{F}_v}(k, l)
\]
then the projections of \(g, h\) on a geodesic of \(\tilde{X}_v\) between \(k\) and \(l\) are at distance smaller than \(M\) one from the other.

(b) There exist \(\lambda > 1\) and an integer \(N \geq 1\) such that for any \(v \in V(T)\), for any \(\alpha, \beta \in V(T)\) with \(d_T(\alpha, v) = d_T(\beta, v) = N\) and \(d_T(\alpha, \beta) = 2N\), if \(x, y \in \tilde{X}_v\) are two exceptional vertices such that \(\tilde{\jmath}_{[v, \alpha]}(x), \tilde{\jmath}_{[v, \alpha]}(y), \tilde{\jmath}_{[v, \beta]}(x)\) and \(\tilde{\jmath}_{[v, \beta]}(y)\) are all well-defined then
\[
\max(d_{\mathfrak{F}_v}(\tilde{\jmath}_{[v, \alpha]}(x), \tilde{\jmath}_{[v, \alpha]}(y)), d_{\mathfrak{F}_v}(\tilde{\jmath}_{[v, \beta]}(x), \tilde{\jmath}_{[v, \beta]}(y))) \geq \lambda d_{\mathfrak{F}_v}(x, y).
\]

Then the fundamental group of \(G\) is strongly hyperbolic relative to a family composed of:

- exactly one representative in each finite parabolic orbit,
- a free extension of exactly one representative in each infinite parabolic orbit.

1.1. Plan of the paper: The results above are consequences of Theorems 3.8 and 4.5 about the behavior of quasi geodesics in trees of hyperbolic spaces. Section 2 contains the basis, from quasi isometries to the “hallways-flare” property. Section 3 deals with the approximation of quasi geodesics in the particular case where all the attaching-maps of the considered tree of hyperbolic spaces are quasi isometries. Section 4 contains the adaptations to the general case. The important notions appearing in these two sections are the corridors in Section 3, and the generalized corridors in Section 4. These two sections appeal to two important Propositions whose proofs are delayed: Proposition 3.9 is proved in Section 7; Proposition 3.10 is proved in Section 8 whereas its adaptation to generalized corridors (Proposition 4.6) is dealt with in subsection 8.6. Section 5 presents the results about the hyperbolicity and the weak relative hyperbolicity whereas Section 6
deals with the consequences about the strong relative hyperbolicity of graphs of strongly relatively hyperbolic groups. This last section contains another proposition whose proof is postponed to subsection 8.7. Subsection 6.1 contains the proofs of the above statements and some additional group-theoretic corollaries.

2. Preliminaries

If \((X, d)\) is a metric space with distance function \(d\), and \(x\) a point in \(X\), we set \(B_x(r) = \{y \in X ; d(x, y) \leq r\}\). If \(A\) and \(B\) are any two subsets of \((X, d)\), \(d'(A, B) = \inf_{x \in A, y \in B} d(x, y)\). We set also \(\mathcal{N}_d^r(A) = \{x \in X ; d'(x, A) \leq r\}\) and \(d''(A, B) = \sup\{r \geq 0 ; A \subset \mathcal{N}_d^r(B)\}\) and \(B \subset \mathcal{N}_d^r(A)\} is then the usual Hausdorff distance between \(A\) and \(B\). Finally, \(\text{diam}_X(A)\) stands for the diameter of \(A\): \(\text{diam}_X(A) = \sup\{d(x, y) ; (x, y) \in A \times A\}\).

2.1. Quasi isometries, quasi geodesics and hyperbolic spaces. A \((\lambda, \mu)\)-quasi isometric embedding from \((X_1, d_1)\) to \((X_2, d_2)\) is a map \(f : X_1 \to X_2\) such that, for any \(x, y\) in \(X_1\):

\[
\frac{1}{\lambda} d_1(x, y) - \mu \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \mu
\]

A \((\lambda, \mu)\)-quasi isometry \(f : (X_1, d_1) \to (X_2, d_2)\) is a \((\lambda, \mu)\)-quasi isometric embedding such that for any \(y \in X_2\) there exists \(x \in X_1\) with \(d_2(f(x), y) \leq \mu\).

A \((\lambda, \mu)\)-quasi geodesic in a metric space \((X, d)\) is the image of an interval of the real line under a \((\lambda, \mu)\)-quasi isometric embedding.

Since a quasi isometric embedding is not necessarily a continuous map, a quasi geodesic as defined above is not a path in the usual sense, but a chain where by “chain” we mean an ordered family of points and oriented continuous paths. As for edge-paths in graphs, if \(c\) is a chain or path in a metric space we denote by \(i(c)\) (resp. \(t(c)\)) its initial (resp. terminal) point.

We work with a version of the Gromov hyperbolic spaces which is slightly extended with respect to the most commonly used. We do not require first that they be geodesic, and second that they be proper, that is closed balls are not necessarily compact. Instead of geodesic spaces, we consider quasi geodesic spaces: a metric space \((X, d)\) is a \((r, s)\)-quasi geodesic space if for any two points \(x, y\) in \(X\) there is a \((r, s)\)-quasi geodesic between \(x\) and \(y\). We then denote by \([x, y]\) such a \((r, s)\)-quasi geodesic (and of course in a geodesic space, \([x, y]\) denotes any geodesic between \(x\) and \(y\)). A quasi geodesic metric space is a metric space which is \((r, s)\)-quasi geodesic for some non-negative real constants \(r, s\). The \((r, s)\)-quasi geodesic triangles in a \((r, s)\)-quasi geodesic metric space \((X, d)\) are thin if there exists \(\delta \geq 0\) such that any \((r, s)\)-quasi geodesic triangle in \((X, d)\) is \(\delta\)-thin, that is any side is contained in the \(\delta\)-neighborhood of the union of the two other sides. In this case, \(X\) is a \(\delta\)-hyperbolic space. A metric space \((X, d)\) is a Gromov hyperbolic space if there exists \(\delta \geq 0\) such that \((X, d)\) is a \(\delta\)-hyperbolic space. The slight “generalization” from geodesic to quasi geodesic spaces is only a technical point. But not requiring our spaces to be proper is important in order to deal with relatively hyperbolic groups, the definitions of which involve non-proper metric graphs.

2.2. Trees of spaces. A metric tree is a simplicial tree with all edges isometric to \((0, 1)\). If \(T\) is a metric tree, we denote by \(|\cdot|_T\) the length of a path in \(T\) and by \(d_T\) the associated distance. Definition 2.1 below requires some basic knowledge of [7] about length metrics (Chapter I.3) and the gluing of metric spaces (Chapter I.5): if \((X_\lambda)_{\lambda \in \Lambda}\) is a family of
metric spaces and $\sim$ is an equivalence relation on their disjoint union $\bigsqcup_{\lambda \in \Lambda} X_\lambda$ then there is a natural pseudo-metric on $\bigsqcup_{\lambda \in \Lambda} X_\lambda / \sim$, termed quotient pseudo-metric, such that $\bigsqcup_{\lambda \in \Lambda} X_\lambda / \sim$ is a length space as soon as each $X_\lambda$ is, and the quotient pseudo-metric is a metric. Let us recall that a length-metric is a metric such that the distance between any two points is equal to the infimum of the lengths of the rectifiable curves between these two points. A length-space is a space equipped with a length-metric. The distance between two equivalence classes $\overline{x}, \overline{y}$ for the quotient pseudo-metric on a quotient of a metric space $(X, d)$ is equal to the infimum of the $\sum_{i} d(x_i, y_i)$ over all the chains $(x_1, y_1, x_2, \cdots, x_n, y_n)$ where $x_1 \in \overline{x}$, $y_n \in \overline{y}$ and $y_i \sim x_{i+1}$.

**Definition 2.1.** (compare [3]) A tree of metric spaces $(\tilde{X}, \mathcal{T}, \pi)$ is a metric space $\tilde{X}$ equipped with a continuous map $\pi : \tilde{X} \to \mathcal{T}$ onto a metric tree $\mathcal{T}$ such that:

(a) If $e$ is an edge of $\mathcal{T}$ and $m_e$ is the midpoint of the edge $e$, then:
   - $X_e := \pi^{-1}(m_e)$ is a geodesic metric space for the induced length-metric,
   - $\pi^{-1}(e)$ is isometric to $X_e \times (0,1)$ equipped with the metric product of the induced length-metric on $X_e$ by the metric of the interval.

(b) If $v$ is a vertex of $\mathcal{T}$, then $X_v := \pi^{-1}(v)$ is a geodesic metric space for the induced length-metric.

(c) For any oriented edge $e$ in $E(\mathcal{T})$, let $\chi_e$ be the disjoint union
   $$X_t(e) \sqcup \bigsqcup_{e' \in E(\mathcal{T}), t(e')=t(e)} X_e \times [0,1/2],$$
   where $X_t(e)$ is equipped with the induced length-metric and $X_e \times [0,1/2]$ with the metric product of the induced length-metric on $X_e$ by the metric of the interval.

There exist $a \geq 1$, $b \geq 0$ and for each oriented edge $e$ of $\mathcal{T}$ a $(a, b)$-quasi isometric embedding $j_e : X_e \to X_t(e)$ such that, if $\sim_e$ is the equivalence relation on $\chi_e$ defined by $x \sim_e y$ if and only if:
   - either $x \in X_e \times \{0\}$, $y \in X_e \times \{0\}$ and $j_e(x) = j_e(y)$,
   - or $x \in X_t(e)$ (resp. $y \in X_t(e)$), $y \in X_e \times \{0\}$ (resp. $x \in X_e \times \{0\}$) and $x = j_e(y)$ (resp. $y = j_e(x)$),

if $T_e$ denotes the tree $\mathcal{T}$ subdivided at the midpoints of the edges then, when equipped with the induced length-metric the pre-image under $\pi$ of the star of $t(e)$ in $T_e$ is isometric to the quotient-space $\chi_e / \sim_e$ equipped with the quotient pseudo-metric.

Let $x$ be any point of $\mathcal{T}$. The set $\pi^{-1}(x)$ equipped with the induced length-metric is a stratum.

A tree of hyperbolic spaces is a tree of metric spaces such that there is $\delta \geq 0$ for which the strata are $\delta$-hyperbolic spaces.

See Figure 2.

**Example 2.2.** Consider the mapping-torus group $F_n^\alpha := F_n \rtimes_\alpha \mathbb{Z}$ of an automorphism $\alpha$ of the rank $n$ free group $F_n$. This is the fundamental group of a graph of groups which is a loop: the vertex group $G_v$ and the edge group $G_e$ are $F_n$, the morphism $\tau_{e^{-1}} : F_n \to F_n$ is the identity whereas $\tau_e : F_n \to F_n$ is the automorphism $\alpha$. This group acts cocompactly, isometrically and properly discontinuously on a tree of 0-hyperbolic spaces $(\tilde{X}, \mathcal{T}, \pi)$ defined as follows: Let $\Gamma$ be the Cayley graph of $F_n$ with respect to some basis $\{x_1, \cdots, x_n\}$, hence a tree, and let $\Gamma_\i := \Gamma \times [i, i + 1]$. We denote by $f_\alpha : \Gamma \to \Gamma$
a continuous map which realizes $\alpha$ on $\Gamma$, that is which sends any edge labeled with the generator $x_i$ to the edge-path corresponding to $\alpha(x_i)$. Then $\hat{X} = \bigsqcup_{i \in \mathbb{Z}} \Gamma_i / \sim$, where $(x, i + 1) \sim (y, i + 1)$ if and only if $(x, i + 1) \in \Gamma_i$, $(y, i + 1) \in \Gamma_{i+1}$ and $y = f_\alpha(x)$. The tree $T$ is homeomorphic to the real line $\mathbb{R}$ as a topological space: this is the graph with one vertex at each integer $i$ and one edge for each interval $[i, i+1]$. The map $\pi: \hat{X} \to T$ sends each $\Gamma_i$ to a point so that each stratum is an isomorphic copy of $\Gamma$. The embedding $j_{i,i+1}: X_{[i,i+1]} \to X_i$ is the identity whereas $j_{i,i+1}: X_{[i,i+1]} \to X_{i+1}$ is $f_\alpha$.

Example 2.3. Consider the amalgamated product $F_2 \ast \mathbb{Z} F_2$ with $F_2 = \langle x_1, x_2 \rangle$ and $\mathbb{Z} = \langle x_1 \rangle$. This is the fundamental group of a graph of groups $G$ with one edge $e$ and one edge-group $G_e := \mathbb{Z} = \langle t \rangle$, with two vertices $v, w$ and vertex-groups $G_v := F_2$, $G_w := F_2$ and with morphisms $\iota_{e^{-1}}, \iota_e: \mathbb{Z} \hookrightarrow F_2$ defined by $\iota_{e^{-1}}(t) = \iota_e(t) = x_1$. This group acts cocompactly, isometrically and properly discontinuously on a tree of 0-hyperbolic spaces $(\hat{X}, T, \pi)$ defined as follows. The tree $T$ is the tree of the universal covering of $G$. Over an open edge, $\hat{X}$ is isometric to $\mathbb{R} \times (0,1)$ and the map $\pi: \hat{X} \to T$ sends each $\mathbb{R} \times \{t\}$ to a point, i.e. $X_e = \mathbb{R}$ and the strata over the points interior to the edges are real lines. The strata over the vertices are the copies of a tree $\Gamma$, which is the Cayley-graph of $F_2$ with respect to $\{x_1, x_2\}$. Each attaching-map $j_e$ is the identity onto its image. The domain and the images of $j_e$ are the axis of some conjugates of $x_1 < F_2$.

Remark 2.4. It is obvious from the definition that a tree of metric spaces is a quasi geodesic metric space as defined in Section 2.1. We could also only require that strata be quasi geodesic spaces, instead of geodesic ones.

By definition, each stratum in a tree of metric spaces comes with a distance, termed horizontal distance. A path contained in a stratum is a horizontal path and we will also speak of the horizontal length of a horizontal path. We extend the definition of the horizontal distance to $\hat{X} \times \hat{X}$ by declaring that the horizontal distance between two points which are not in a same stratum is infinite.

2.3. The telescopic metric. A section of a map $\pi: A \to B$ is a map $\sigma: B \to A$ such that $\pi \circ \sigma = \text{Id}_B$. This notion of section is only a set-theoretic notion: unless otherwise specified, we do not require that a section of a continuous map be continuous, of a morphism be a morphism, . . . .

Definition 2.5. Let $(\hat{X}, T, \pi)$ be a tree of metric spaces.
For $v \geq 0$, a $v$-vertical segment (resp. $v$-vertical tree) in $\tilde{X}$ is a section $\sigma_{\omega}$ (resp. $\sigma_T$) of $\pi$ over a geodesic $\omega$ of $T$ (resp. over a subtree $T$ of $T$) which is a $(v+1,v)$-quasi isometric embedding.

The $T$-length $|\omega|_T$ is the vertical length of the $v$-vertical segment $\sigma_{\omega} : \omega \to \tilde{X}$.

If $x$ is a point in $\tilde{X}$ and if $\omega$ is a geodesic of $T$ starting at $\pi(x)$, the notation $\omega x$ denotes the set of points $y \in \tilde{X}$ such that some $v$-vertical segment $s$ with $\pi(s) = \omega$ connects $x$ to $y$.

**Remark 2.6.** With the notations above, observe that in particular any point in $\omega x$ belongs to $\pi^{-1}(t(\omega))$.

By a slight abuse of terminology we will not distinguish a vertical segment or tree, which by definition is a map, from its image in the tree of spaces. Since a section is not necessarily continuous, this image is of course not a segment nor a tree in the usual sense. But if $\omega = e^{\omega}_{i_1} \cdots e^{\omega}_{i_n}$ is a geodesic edge-path then a $v$-vertical segment over $\omega$ can be approximated by a chain of intervals $x_j \times (0,1)$ over the $e^\omega_i$’s and points $y_i$ in the $\pi^{-1}(t(e^{\omega}_{i}))$ (see Definition 2.1), where the Hausdorff distance between the $v$-vertical segment and this chain only depends on $v$. Of course a similar approximation exists in the case of a $v$-vertical tree.

**Definition 2.7.** Let $(\tilde{X}, T, \pi)$ be a tree of metric spaces.

For $v \geq 0$, a $v$-telescopic chain is a chain $p$ in $\tilde{X}$ which satisfies the following properties:

- $\pi(p)$ is an edge-path between two vertices of $T$,
- $p$ is a concatenation $h_0 s_0 \cdots h_j s_j \cdots s_h h_{n+1}$, with $t(s_j) = i(h_{j+1})$ and $t(h_j) = i(s_j)$ for $j = 0 \cdots n$, of horizontal paths $h_j$ in the strata over the vertices of $T$ with non-trivial $v$-vertical segments $s_j$.

**Definition 2.8.** Let $v \geq 0$ and let $p$ be a $v$-telescopic chain in a tree of metric spaces $(\tilde{X}, T, \pi)$.

(a) The vertical length $|p|^v_{\text{vert}}$ of $p$ is the sum of the vertical lengths of the maximal $v$-vertical segments. The horizontal length $|p|^v_{\text{hor}}$ is the sum of the horizontal lengths of the maximal horizontal paths in the complement of the maximal $v$-vertical segments.

(b) The telescopic length $|p|^v_{\text{tel}}$ of a $v$-telescopic chain $p$ is the sum of its horizontal and vertical lengths.

**Definition 2.9.** Let $(\tilde{X}, T, \pi)$ be a tree of metric spaces. For $v \geq 0$, the $v$-telescopic distance $d^v_{\text{tel}}(x, y)$ between two points $x$ and $y$ is the infimum of the telescopic lengths of the $v$-telescopic chains between $x$ and $y$.

**Remark 2.10.** Let $v \geq 0$ and let $p$ be a $v$-telescopic chain. The vertical length of each maximal $v$-vertical segment in $p$ is greater or equal to 1.

Any point in $\tilde{X}$ is at vertical distance smaller than $\frac{1}{2}$ from a stratum over a vertex of $T$. Thus, when dealing with the behavior of quasi geodesics or with the hyperbolicity of $\tilde{X}$ there is no harm in requiring that telescopic chains begin and end at strata over vertices of $T$, as was done in Definition 2.7.

For the sake of simplification, we will often forget the exponents in the vertical, horizontal and telescopic lengths, unless some ambiguity might exist.

**Lemma 2.11.** Let $(\tilde{X}, T, \pi)$ be a tree of hyperbolic spaces.

(a) For any $v \geq 0$ there exist $\lambda_+ \geq 1$, $\mu \geq 0$ such that, if $\omega_0$ and $\omega_1$ are any two $v$-vertical segments, with initial (resp. terminal) points $x_0, x_1$ (resp. $y_0, y_1$) and such that $\pi(\omega_0) = \pi(\omega_1) = [a, b]$ then:
\[ \frac{1}{\lambda^+(a,b)} d_{\text{hor}}(x_0, x_1) - \mu \leq d_{\text{hor}}(y_0, y_1) \leq \lambda^+(a,b) d_{\text{hor}}(x_0, x_1) + \mu \]

The constants \( \lambda^+, \mu \) will be referred to as the constants of quasi isometry.

(b) \( \lim_{n \to +\infty} d_{\text{hor}}(x_0, x_n) = +\infty \iff \lim_{n \to +\infty} d_{\text{tel}}(x_0, x_n) = +\infty \) whenever \((x_n)_{n \in \mathbb{Z}^+}\) is a sequence of points in some stratum.

(c) For any \( v, v' \geq 0 \), there exists \( A \geq 1, B \geq 0 \) such that the identity-map from \((\tilde{X}, d_{\text{tel}}')\) to \((\tilde{X}, d_{\text{tel}}')\) is a \((A, B)\)-quasi isometry.

(d) For any \( d, v \geq 0 \) there exists \( C \geq 0 \) increasing with both \( d \) and \( v \) such that for any \( \alpha, \beta \in \mathcal{T} \) with \( d_T(\alpha, \beta) = d \), for any \( x, y, z \in X_\alpha \) with \( z \in [x, y] \), whenever \( x', y', z' \in X_\beta \) are the endpoints of \( v \)-vertical segments starting respectively at \( x, y \) and \( z \), then \( z' \in N^C_{\text{hor}}([x', y']) \).

(e) For any \( v, w \geq 0 \), there is \( D \geq 0 \) such that if \( s \) is a \( v \)-vertical segment, then \( s \) is a \((D, D)\)-quasi geodesic for the \( w \)-telescopic distance.

Proof: Item (a) is a straightforward consequence of the definition of a vertical segment. Items (b) and (c) are consequences of the existence of the constants of quasi isometry given by the first item. Item (d) amounts to saying that the image of a geodesic under a \((a, b)\)-quasi isometric embedding is \(C(a, b)\)-close to any geodesic between the images of the endpoints. This is a well-known assertion, see for instance [9]. Like Item (a), Item (e) is checked by a straightforward computation. \( \square \)

Remark 2.12. Throughout all the text, the constants appearing in each lemma, corollary or proposition will be denoted by \( C, D, \cdots \) and thereafter they will be referred to by the same letter with the number of the lemma, corollary or proposition in subscript. For instance, if Lemma 3.4 introduces the constants \( C \) and \( D \) then for referring afterwards to these constants we will write \( C_{3,4} \) and \( D_{3,4} \).

The following lemma relates the telescopic metric to the original one. In the sequel, it allows one to indistinctly use the original metric or a telescopic metric \( d_{\text{tel}}' \).

Lemma 2.13. Let \((\tilde{X}, \mathcal{T}, \pi)\) be a tree of hyperbolic spaces.

(a) For any \( v \geq 0 \), there exist \( C \geq 1, D \geq 0 \) such that the identity-map from \( \tilde{X} \) equipped with its original metric to \( \tilde{X} \) equipped with the \( v \)-telescopic metric is a \((C, D)\)-quasi isometry.

(b) For any \( r \geq 1, s, v \geq 0 \), there exist \( C' \geq 1 \) and \( D' \geq 0 \) such that any \( v \)-telescopic chain which is a \((r, s)\)-quasi geodesic of \((\tilde{X}, d_{\text{tel}}')\) is a \((C', D')\)-quasi geodesic of \( \tilde{X} \) equipped with its original metric.

(c) For any \( r \geq 1, s, v \geq 0 \), there exist \( E \geq 1, F \geq 0 \) such that, given any \((r, s)\)-quasi geodesic \( g \) of \( \tilde{X} \) equipped with the original metric, there is a \( v \)-telescopic \((E, F)\)-quasi geodesic \( G \) whose Hausdorff distance from \( g \) is bounded above by \( E \).

Observe that, once Item (a) has been proved Item (b) is straightforward and it is useless in Item (c) to precise what is the Hausdorff distance we refer to: the one coming the original metric or the telescopic one. Indeed, by Item (a) the conclusion of Item (c) holds for the former if and only if it holds for the latter.

Proof: Consider a geodesic \( g \) of \( \tilde{X} \) equipped with its original metric. Let us define two kinds of modifications:
Let $g' \subset g$ be a maximal subpath with both endpoints in a same stratum over a vertex $v$ of $T$ and such that $\pi(g') \subset e$ for some edge $e$ incident to $v$. Then we substitute $g'$ by a horizontal geodesic in $X_v$ between its endpoints.

Let $g' \subset g$ be a subpath of $g$ with $\pi(g') = e$, $\pi(i(g')) = i(e)$, $\pi(t(g')) = t(e)$. Then we substitute $g'$ by the concatenation of a 0-vertical segment $s$ from $i(g')$ to $t(i(e))$ with a horizontal geodesic from $t(s)$ to $t(g')$.

Let $G$ be the path obtained after these substitutions. This is a 0-telescopic chain. In both kinds of substitutions we add a horizontal geodesic $h$ in some stratum $X_v$. Thanks to the fact that strata are Gromov hyperbolic and edge-spaces are $(a, b)$-quasi isometrically embedded in the vertex-spaces, there is a bound (not depending on $g$ nor $G$) on the horizontal Hausdorff distance between $\beta, \gamma$ such that, for any subpath $G'$ of $G$ there is a subpath $g'$ of $g$ which is at Hausdorff distance smaller than $E$ from $G'$ and such that $|G'_{tel}| \leq E|g'| + F$. From this we get $d_{tel}^g \leq Ed_{\pi} + F$. Since $d_{\pi} \leq d_{tel}^g$ is obvious (a 0-telescopic chain is a particular kind of path in the usual sense) we get that the identity realizes a quasi isometry between the 0-telescopic metric and the original metric. Since all telescopic metrics are quasi isometric (see Lemma 2.11, Item (c)) we get Item (a). Item (b) is then obvious and Item (c) is proven in the same way as above by considering a quasi geodesic $g$ instead of a geodesic. □

2.4. **Exponential separation of vertical segments, hallways-flare property.** The term of “hallways-flare property” was introduced in [3]; it designated the main property introduced by the authors for the hyperbolicity of a graph of quasi isometrically embedded hyperbolic groups. Although the presentation here is very different, we use this same denomination for our central property given in Definition 2.14 below and invite the reader to compare with the “hallways-flare property” of [3].

**Definition 2.14.** (compare [3])

A tree of hyperbolic spaces $(\tilde{X}, T, \pi)$ satisfies the **hallways-flare property** if for any $v \geq 0$ there exist $\lambda > 1$ and positive integers $t_0, M$ such that, for any $\alpha \in T$, for any two $\beta, \gamma \in \partial B_2(t_0)$ with $d_T(\beta, \gamma) = 2t_0$, any two $v$-vertical segments $s_0, s_1$ over $[\beta, \gamma]$ with $d_{hor}(s_0 \cap X_\alpha, s_1 \cap X_\alpha) \geq M$ satisfy:

$$
\max(d_{hor}(s_0 \cap X_\beta, s_1 \cap X_\beta), d_{hor}(s_0 \cap X_\gamma, s_1 \cap X_\gamma)) \geq \lambda d_{hor}(s_0 \cap X_\alpha, s_1 \cap X_\alpha)
$$

Figure 3 illustrates a situation where the above property is satisfied (left picture) and one where it is not (right picture). See also Figure 1.

We will sometimes say that the $v$-vertical segments are **exponentially separated.** The constants $\lambda, M, t_0$ will be referred to as the **constants of hyperbolicity.**

The hallways-flare property requires the exponential separation of the $v$-vertical segments for any $v \geq 0$. It suffices in fact that it be satisfied for some $v$ sufficiently large enough as we are now going to check (see Lemma 2.18).

**Definition 2.15.** Let $\tilde{X}$ be a tree of hyperbolic spaces and let $S$ be a horizontal subset which is quasi convex in its stratum, for the horizontal metric. If $x$ is any point in $\tilde{X}$ then a horizontal quasi projection of $x$ to $S$, denoted by $P^\text{hor}_S(x)$, is any point $y$ in $S$ such that $d_{hor}(x, y) < d^\text{hor}_S(x, S) + 1$.

If $x$ and $S$ do not belong to a same stratum, such a horizontal quasi projection does not exist, the horizontal distance $d_{hor}(x, y)$ being infinite for any $y \in S$. 

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Lemma 2.16. Let $\delta \geq 0$ and let $(\tilde{X}, T, \pi)$ be a tree of $\delta$-hyperbolic spaces. There exists $C \geq 0$ such that if $v \geq C$, if $e$ is an edge of $T$ and if $h$ is a horizontal geodesic in $X_{t(e)}$, then:

- If no $v$-vertical segment starting at $h$ can be defined over $e$, then
  \[ \text{diam}_{X_{t(e)}}(P_{h}^{\text{hor}}(j_{e}(X_{e}))) \leq 2\delta \]

- If $v$-vertical segments can be defined over $e$ starting at the initial and terminal points of $h$, then $v$-vertical segments can be defined over $e$ starting at any point in $h$.

We recall that for each oriented edge $e$, $j_{e}$ denotes the $(a, b)$-quasi isometric embedding of the edge-space $X_{e}$ into the vertex-space $X_{t(e)}$ associated to the given tree of spaces.

Example 2.17. Let $\tilde{X}$ be a tree of 0-hyperbolic spaces (i.e. of $\mathbb{R}$-trees - see Examples 2.2 and 2.3). Then one can set $C_{2.16} := 0$. Indeed, the 0-hyperbolicity of the strata implies that, if $x$ and $y$ are two points in $j(X_{e})$ then the whole geodesic of $X_{t(e)}$ between $x$ and $y$ is contained in $j(X_{e})$. Thus, if $h$ is a horizontal geodesic in $X_{t(e)}$ and $e$ an edge of $T$ such that no 0-vertical segment starting from $h$ is defined over $e$, then any horizontal geodesic of $X_{t(e)}$ between two points $x, y$ of $j(X_{e})$ is disjoint from $h$. This readily implies $\text{diam}_{X_{t(e)}}(P_{h}^{\text{hor}}(j_{e}(X_{e}))) = 0$.

Problems occur as soon as one deals with trees of $\delta$-hyperbolic spaces with $\delta > 0$. Then the constants $\delta$, $a$ and $b$ come into play. See Figure 4 and the proof below.
points \( x, y \in \partial(X_e) \), any horizontal geodesic \([x, y]\) lies in the horizontal \( e \)-neighborhood of \( \partial(X_e) \). Choose \( v > 2\delta + e \). Assuming that no \( v \)-vertical segment starting at \( h \) can be defined over \( e \), since horizontal geodesic rectangles are \( 2\delta \)-thin, we get \([x, y] \cap N_{\text{hor}}(h) = \emptyset\) for any two points \( x, y \in \partial(X_e) \) and any horizontal geodesic \([x, y]\). The conclusion of the first item follows by the \( 2\delta \)-thinness of the geodesic rectangles. The second item of the lemma is proved in the same way, details are left to the reader. \(\square\)

**Lemma 2.18.** Let \((\tilde{X}, \mathcal{T}, \pi)\) be a tree of hyperbolic spaces. If \( v \geq C_{2.16} \) is such that the \( v \)-vertical segments of \( \tilde{X} \) are exponentially separated with constants of hyperbolicity \( \lambda_v > 1 \), \( M_v, t_0 \geq 0 \) then for any \( w \geq 0 \), the \( w \)-vertical segments are exponentially separated, with constants of hyperbolicity \( \lambda_w > 1 \), \( M_w \geq 0 \) and \( t_0 \).

**Proof:** The statement is a tautology if \( w \leq v \). We thus assume \( w \geq v \). Consider \( \alpha, \beta, \gamma \) in \( \mathcal{T} \) with \( \alpha \in [\beta, \gamma] \) and \( d_{\mathcal{T}}(\alpha, \beta) = d_{\mathcal{T}}(\alpha, \gamma) = t_0 \). Consider two \( w \)-vertical segments \( S_0, S_1 \) over \([\beta, \gamma]\) with \( d_{\text{hor}}(x_0, x_1) \geq M \), where \( x_i = S_i \cap X_\alpha \) and \( M > M_v \). We distinguish two cases:

- there exist \( v \)-vertical segments \( s_0, s_1 \) passing through \( x_0, x_1 \) and defined over \([\beta, \gamma]\).
  From Item (a) of Lemma 2.11, each endpoint of the \( s_i \)'s is at bounded horizontal distance from an endpoint of a \( S_i \), where the upper-bound only depends on \( w, t_0 \) and the constants of quasi isometry. Thus choosing \( M \) sufficiently large enough with respect to \( w \) gives the desired inequality between \( d_{\text{hor}}(x_0, x_1) \) and \( \max(d_{\text{hor}}(S_0 \cap X_\beta, S_1 \cap X_\beta), d_{\text{hor}}(S_0 \cap X_\gamma, S_1 \cap X_\gamma)) \).
- the other case: since \( v \) has been chosen greater than \( C_{2.16} \), there is some stratum \( X_\mu, \mu \in [\beta, \gamma] \) such that \( d_{\text{hor}}(S_0 \cap X_\mu, S_1 \cap X_\mu) \) is bounded above by a constant depending on \( w, \delta, t_0 \) and the constants of quasi isometry. By Item (a) of Lemma 2.11, we get an upper-bound on \( d_{\text{hor}}(x_0, x_1) \). Setting \( M \) greater that this upper-bound, we get the lemma.

\(\square\)

We end this section by a very general and easy lemma about the constants of hyperbolicity.

**Lemma 2.19.** Let \((\tilde{X}, \mathcal{T})\) be a tree of hyperbolic spaces satisfying the hallways-flare property.

(a) The constants of hyperbolicity and quasi isometry can be chosen arbitrarily large enough.

(b) For any \( v \geq C_{2.16} \) such that the \( v \)-vertical segments are exponentially separated, for any constants of hyperbolicity \( \lambda, M, t_0 \) such that \( M \) is sufficiently large enough, there exists \( C \geq 0 \) such that for any \( \alpha \in \mathcal{T} \), for any \( \beta, \gamma \in \partial B_{t_0}(\alpha) \) with \( \alpha \in [\beta, \gamma] \), for any two \( v \)-vertical segments \( s_0, s_1 \) over \([\beta, \gamma]\) such that \( d_{\text{hor}}(x_0, x_1) \geq M \) where \( x_i = s_i \cap X_\alpha \), if the endpoints \( y_0, y_1 \) of \( s_0, s_1 \) in \( X_\beta \) (resp. in \( X_\gamma \)) satisfy:

\[
\frac{1}{\lambda} d_{\text{hor}}(x_0, x_1) < d_{\text{hor}}(y_0, y_1),
\]

then, for any \( n \geq 1 \), for any \( \mathcal{T} \)-geodesic \( \omega \) starting at \( \alpha \) with \([\alpha, \beta] \subset \omega \) (resp. \([\alpha, \gamma] \subset \omega \)) and \(|\omega|_{\mathcal{T}} \geq C + nt_0\):

\[
d_{\text{hor}}^i(\omega x, \omega y) \geq \lambda^n d_{\text{hor}}^i(x, y).
\]

### 3. Approximation of quasi geodesics: a “simple” case

From a group-theoretical point of view, the case treated in this section allows one to deal with semi-direct products of (relatively) hyperbolic groups with free groups but not
with HNN-extensions and amalgamated products along proper subgroups. For this we need the similar, but more general, result of Section 4.

Beware that the corridors (and further the generalized corridors) defined below are not the hallways of [3]. The reason is that we are interested in exhibiting quasi convex subsets of our trees of hyperbolic spaces and the hallways of [3], in general, are not quasi convex.

**Definition 3.1.** Let $$(\tilde{X}, \mathcal{T}, \pi)$$ be a tree of hyperbolic spaces, and let $v \geq 0$.

A $v$-vertical tree $\sigma: \mathcal{T} \to \mathcal{T}$ is maximal if and only if there exists no $v$-vertical tree $\sigma': \mathcal{T}' \to \mathcal{T}$ such that $\mathcal{T} \subset \mathcal{T}'$, $T \neq T'$ and $\sigma'|_T = \sigma$.

A $v$-corridor $\mathcal{C}$ is a union of horizontal geodesics which satisfies the following properties:

(a) For each $\alpha \in \mathcal{T}$, $\mathcal{C} \cap X_\alpha$ either is empty or is equal to a horizontal geodesic.

(b) There exists a subtree $\mathcal{T}$ of $\mathcal{T}$ such that $\mathcal{C} = \bigcup_{\alpha \in \mathcal{T}} h_\alpha$ where each $h_\alpha = \mathcal{C} \cap X_\alpha$ is a horizontal geodesic.

(c) There exist two maximal $v$-vertical trees $\sigma_1, \sigma_2: \mathcal{T} \to \tilde{X}$ such that each $h_\alpha$ connects a point of $\sigma_1(T)$ to a point of $\sigma_2(T)$.

The subsets $\sigma_1(T)$ and $\sigma_2(T)$ are the vertical boundaries of the $v$-corridor $\mathcal{C}$.

**Example 3.2.** Consider the tree of spaces associated to the mapping-torus $\mathbb{F}_n^\alpha$ of a free group automorphism as described in Example 2.2. Then any horizontal geodesic $h$ is contained in a 0-corridor $\mathcal{C}$ with $\pi(\mathcal{C}) = \mathcal{T}$. If $h$ connects two vertices of the Cayley graph of $\mathbb{F}_n$, then the vertical boundaries of $\mathcal{C}$ are the $\alpha$-orbits of the endpoints of $h$.

**Example 3.3.** Consider the tree of spaces of Example 2.3. Given a horizontal geodesic $h$, there will be different kinds of 0-corridors containing $h$, depending on the position of $h$ in the Cayley graph of $\mathbb{F}_2$. For simplicity, the geodesics we consider always have their endpoints at vertices of this Cayley graph.

If $h$ is contained in the axis of a conjugate of $x_1$ (for instance $h = x_1^2$), then $h$ is contained in a 0-corridor $\mathcal{C}$ such that $\pi(\mathcal{C})$ is a bi-infinite geodesic of $\mathcal{T}$. Each horizontal geodesic in $\mathcal{C}$ reads the same word, a power of $x_1^{\pm 1}$.

For any other horizontal geodesic $h$, for instance $h = x_2 x_1^2 x_2$, the unique 0-corridor $\mathcal{C}$ containing $h$ is equal to $h$.

**Remark 3.4.** Let $$(\tilde{X}, \mathcal{T}, \pi)$$ be a tree of hyperbolic spaces the attaching-maps of which are all quasi isometries (and not only quasi isometric embeddings). Then, as soon as $v \geq C_{2,16}$, given any two points $x, y$ in $\tilde{X}$ there is a $v$-corridor $\mathcal{C}$ whose vertical boundaries pass through $x$ and $y$. Moreover $\pi(\mathcal{C}) = \mathcal{T}$.

**Definition 3.5.** Let $\mathcal{C}$ be a union of horizontal geodesics in a tree of hyperbolic spaces $$(\tilde{X}, \mathcal{T})$$.

Assume that for each stratum $X_\alpha$ the intersection $\mathcal{C} \cap X_\alpha$ either is empty or is equal to a horizontal geodesic.

If $x$ is a point in a stratum $X_\alpha$, then $P^\text{hor}_{\mathcal{C} \cap X_\alpha}(x)$ stands for the horizontal quasi projection $P^\text{hor}_{\mathcal{C} \cap X_\alpha}(x)$ of $x$ to $\mathcal{C}$ (see Definition 2.15).

In the definition above, for instance $\mathcal{C}$ might be a corridor.

Before stating Lemma 3.6 below, we would like to insist on two points:

- The horizontal quasi projection $P^\text{hor}_{\mathcal{C}}$ is a projection in the strata which only refers to the horizontal metric defined on each stratum.

- Item (b) does not tell anything about the behavior of the telescopic (quasi)-geodesics in a tree of hyperbolic spaces. It only allows one to consider a corridor as a quasi geodesic telescopic metric space.
Lemma 3.6. Let \((\hat{X}, T, \pi)\) be a tree of hyperbolic spaces. For any \(v \geq C_{2.16}\) there exists \(C \geq v\) such that, if \(C\) is a \(v\)-corridor in \(\hat{X}\) then:

(a) For any \(v\)-vertical segment \(s\), \(P_C^{\text{hor}}(s)\) is a \(C\)-vertical segment.

(b) For any \(w \geq C\), \(C\) equipped with the length-metric induced by the \(w\)-telescopic metric on \(\hat{X}\) is a quasi geodesic metric space, denoted by \((C, d_C^{w})\).

Figure 5 illustrates Lemma 3.6.

Proof: If \(\sigma : \omega \to \hat{X}\) is the section of \(\pi\) such that \(s = \sigma(\omega)\) then \(P_C^{\text{hor}}(s)\) is the image of \(\omega\) under the map \(P_C^{\text{hor}} \circ \sigma\). This map is a section of \(\pi\) since the horizontal quasi projection \(P_C^{\text{hor}}\) is a projection in each stratum. We want to prove the existence of \(a \geq 1, b \geq 0\) independent of \(\omega\) such that \(P_C^{\text{hor}} \circ \sigma\) is a \((a, b)\)-quasi isometric embedding of \(\omega\) into \(\hat{X}\).

Assume that \(\omega\) is a single edge. Since \(v \geq C_{2.16}\) and since \(C\) is a \(v\)-corridor, if it is defined over \(\omega\) then \(v\)-vertical segments can be defined over \(\omega\) starting at each point of \(C \cap X_{i(\omega)}\).

Let \(\sigma_0 : \omega \to \hat{X}\) be such a \(v\)-vertical segment starting at \(P_C(\sigma(t(\omega)))\). By Items (a) and (c) of Lemma 2.11, \(d_{\text{hor}}(P_C(\sigma(t(\omega))), \sigma_0(P_C(\sigma(i(\omega))))\) is bounded above by a constant. Thanks to Item (b) of Lemma 2.11, this proves Item (a) of the current lemma. Item (b) is now easy.

Definition 3.7. Let \((\hat{X}, T, \pi)\) be a tree of metric spaces, let \(v \geq 0\) and let \(\sigma_1, \sigma_2 : T \to \hat{X}\) be two maximal \(v\)-vertical trees in \(\hat{X}\). A diagonal between \(\sigma_1(T)\) and \(\sigma_2(T)\) is any horizontal geodesic \(D\) with \(\pi(D) \in V(T)\), with endpoints in \(\sigma_1(T)\) and \(\sigma_2(T)\) and such that any other horizontal geodesic \(h\) satisfying these two properties also satisfies \(|D|_{\text{hor}} \leq |h|_{\text{hor}}\).

The diagonal distance between two maximal \(v\)-vertical trees \(\sigma_1, \sigma_2 : T \to \hat{X}\) is the horizontal length of any diagonal between \(\sigma_1(T)\) and \(\sigma_2(T)\).

In other words, a diagonal is a horizontal geodesic which minimizes the horizontal distance between two maximal vertical trees passing through its endpoints. See Figure 6. It might happen that a diagonal be reduced to a single point, in which case the diagonal distance between the two vertical trees considered vanishes (hence the diagonal distance is in fact a pseudo-distance).

Before the statement of Theorem 3.8, we would like to point out that this is a theorem about trees of hyperbolic spaces whose attaching-maps are quasi isometries, and not only quasi isometric embeddings. The main feature of this theorem is to approximate quasi geodesics of \(\hat{X}\) by some kind of “canonical” quasi geodesics, which in particular are
telescopic chains. We stated the results of this theorem by using the original metric of $\tilde{X}$. But by Lemma 2.13 these results also hold for any telescopic metric and it is easily seen that they also hold for $(C, d^\text{tel}_v)$ when this makes sense.

**Theorem 3.8.** Let $\tilde{X}$ be a tree of hyperbolic spaces which satisfies the hallways-flare property. Assume that each attaching-map from an edge-space into a vertex-space is a quasi isometry. Then:

- for any $v \geq C_{2,16}$, there is $E \geq v$,
- for any $L > 0$ greater than some critical constant and for any $v \geq C_{2,16}$ there exists $D \geq 1$,
- for any $L > 0$ greater than some critical constant, for any $v \geq C_{2,16}$, for any $a \geq 1$ and $b \geq 0$ there exists $C \geq 0$,

such that for any pair of distinct points $x, y$ in $\tilde{X}$, for any $v$-corridor $C$ whose vertical boundaries pass through $x$ and $y$, there is a $E$-telescopic chain $P$ in $C$ satisfying the following properties:

(a) This is a $(D, D)$-quasi geodesic of $(\tilde{X}, d_{\tilde{X}})$.

(b) At the exception of at most one, each maximal horizontal path in $P$ is a diagonal with horizontal length greater or equal to $L$ whereas the last maximal horizontal path has horizontal length less or equal to $L$.

(c) For any $(a, b)$-quasi geodesic $g$ in $\tilde{X}$ with endpoints $x$ and $y$, the Hausdorff distance in $(\tilde{X}, d_{\tilde{X}})$ between $g$ and $P$ is bounded above by $C$.

(d) Let $P'$ be the closed complement in $P$ of its first and last maximal vertical segments. Let $g'$ be any $(a, b)$-quasi geodesic in $\tilde{X}$ whose endpoints lie in the vertical boundaries of $C$. Let $s_0, s_1$ be the $v$-vertical segments in the vertical boundaries of $C$ from the endpoints of $P'$ to the endpoints of $g'$. Then $g'$ is at Hausdorff distance smaller than $C$, in $(\tilde{X}, d_{\tilde{X}})$, from the concatenation of $P'$ with $s_0$ and $s_1$.

For proving this theorem, we need two important propositions which we state now but the proofs of which are postponed for a while. For the understanding of Proposition 3.9, let us recall that we proved in Lemma 3.6 that a corridor $C$ in a tree of hyperbolic spaces $\tilde{X}$ becomes a quasi geodesic metric space when equipped with the length-metric induced by the $v$-telescopic metric on $\tilde{X}$, as soon as $v$ is sufficiently large enough. This quasi geodesic metric space is denoted by $(C, d^w_{\text{tel}})$. This notation does not mean that we proved that $C$ is a tree of hyperbolic spaces.

**Proposition 3.9.** Let $\tilde{X}$ be a tree of hyperbolic spaces which satisfies the hallways-flare property.

For any $v \geq C_{2,16}$, there is $D \geq v$ such that for any $w \geq D$, $L > 0$, $a \geq 1$ and $b \geq 0$ there exists $C \geq 0$ such that if $C$ is a $v$-corridor in $\tilde{X}$, if $L$ is the horizontal length of some horizontal geodesic $[x, y]$ in $C$, if $g$ is a $(a, b)$-quasi geodesic of $(C, d^w_{\text{tel}})$ from a $w$-vertical
tree through $x$ to a $w$-vertical tree though $y$, then $g$ is contained in the $C$-neighborhood of the union of the $w$-vertical segments connecting its endpoints to $x$ and $y$. The constant $C$ is increasing with $L$ as soon as $L$ is greater than some critical constant.

See Section 7 for a proof, and Figure 7 for an illustration.

![Figure 7.](image)

**Proposition 3.10.** Let $X$ be a tree of hyperbolic spaces which satisfies the hallways-flare property and the attaching-maps of which are quasi isometries. For any $v \geq C_{2,16}$, $a \geq 1$ and $b \geq 0$ there exists $C \geq 0$ such that, if $g$ is a $(a,b)$-quasi geodesic in $X$ and if $C$ is a $v$-corridor the vertical boundaries of which pass through the endpoints of $g$ then

$$g \subset N^v_{X}(C).$$

See Section 8 for a proof.

We will also need the following two much easier statements.

**Lemma 3.11.** Let $X$ be a tree of hyperbolic spaces. There exists $C \geq 0$ such that for any $v \geq 0$, for any $v$-corridor $C$ in $X$, for any two points $x, y$ in a same stratum intersected by $C$, $d_{\text{hor}}(P^h(x), P^h(y)) \leq d_{\text{hor}}(x, y) + C$. The same inequality holds for the horizontal quasi projections of $x$ and $y$ to the image of the embedding of an edge-space into a vertex-space.

**Proof:** Since there is $\delta \geq 0$ such that strata are $\delta$-hyperbolic spaces for the horizontal metric and the subspaces to which one projects are quasi convex subsets of their stratum for this horizontal metric, this is a consequence of [9], Corollary 2.2.

**Lemma 3.12.** Let $X$ be a tree of hyperbolic spaces. For any $v \geq C_{2,16}$, there is $D \geq v$ and for any $v \geq C_{2,16}$, $a \geq 1$ and $b, v \geq 0$ there is $C \geq 1$ such that for any $(a,b)$-quasi geodesic $g$ of $X$ and for any $v$-corridor $C$, if $g \subset N^v_{hor}(C)$ then any horizontal quasi projection $P^h_C(g)$ is a $D$-telescopic $(C, C)$-quasi geodesic of $(C, d^h_{\text{tel}})$.

**Proof:** By Lemma 2.13 we can assume that $g$ is a $v$-telescopic chain. By Lemma 3.6 $P^h_C(g)$ is a $C_{3,6}(v)$-telescopic chain. Let us consider any two points $x, y$ in $G = P^h_C(g)$. There are $r$-close to two points $x', y'$ in $g$. We denote by $g_{x'y'}$ the subpath of $g$ between these two points and by $G_{xy}$ the subset of $G$ between $x$ and $y$. Since we now consider the $C_{3,6}(v)$-telescopic distance, $|G_{xy}|_{\text{vert}} = |g_{x'y'}|_{\text{vert}}$. From Lemma 3.11 and since any two maximal horizontal paths in $G$ are separated by a vertical segment of vertical length at least 1, we then get $|G_{xy}|_{\text{vert}} \leq 2|C_{3,11}|g_{x'y'}|_{\text{tel}}$. Since $g$ is a $v$-telescopic $(a,b)$-quasi geodesic, $|g_{x'y'}|_{\text{tel}} \leq ad^v_{\text{tel}}(x', y') + b$. But $d^v_{\text{tel}}(x', y') \leq 2r + d^v_{\text{tel}}(x, y)$. Therefore:

$$|G_{xy}|_{\text{vert}} \leq 2C_{3,11}(a(2r + d^v_{\text{tel}}(x, y)) + b).$$

Since all telescopic distances are quasi isometric (Item (c) of Lemma 2.11), we so get the right inequality for the quasi geodesicity of $P^h_C(g)$. We leave the reader work out the
similar proof of the left inequality.

Proof of Theorem 3.8: Let $\hat{X}$ be a tree of hyperbolic spaces which satisfies the hallways-flare property and such that each attaching-map from an edge-space into a vertex-space is a quasi isometry. Let $v \geq C_{2.16}$. Let $x, y$ be any two distinct points in $\hat{X}$ and let $C$ be any $v$-corridor whose vertical boundaries pass through $x$ and $y$. Since the attaching maps of the tree of hyperbolic spaces are all quasi isometries, $\pi(C) = T$. From Lemmas 2.18 and 3.6, $(C, d_{tel}^{\alpha})$ is a quasi geodesic metric space and the $C_{3,6}$-vertical segments are exponentially separated. From Item (b) of Lemma 2.19, this implies in particular that the endpoints of any diagonal with horizontal length greater than some constant $M$ are exponentially separated in all the directions of $T$ outside a region with vertical width bounded by $2C_{2.19}$.

Let $L \geq M$. Consider a diagonal $h_0$ with horizontal length $L$ from a vertical boundary $B_0$ of $C$ to some maximal $C_{3,6}$-vertical tree $T_0$ in $C$. Then another diagonal $h_1$ from $T_0$ to another maximal $C_{3,6}$-vertical tree $T_1$, and so on until arriving at a maximal $C_{3,6}$-vertical tree $T_r$ which is at diagonal distance smaller than $L$ from the other $v$-vertical boundary tree $B_1$ of $C$. Then the concatenation of:

- the diagonals $h_0, h_1, \ldots, h_r$,
- the $C_{3,6}$-vertical segments in $T_0, T_1, \ldots, T_{r-1}$ between the endpoints of the $h_i$’s,
- a horizontal geodesic $h_{r+1}$ with $|h_{r+1}|_{hor} \leq M$ between $T_r$ and $B_1$ which is closest to $h_r$ with respect to the vertical distance,
- the $C_{3,6}$-vertical segment in $T_r$ between $h_r$ and $h_{r+1}$,

gives to us the telescopic chain denoted by $\mathcal{P}'$ in the last item of Theorem 3.8, whereas of course the concatenation with the $v$-vertical segments in $B_0$ and $B_1$ between the horizontal geodesics $h_0$ and $h_{r+1}$ and the points $x$ and $y$ gives the announced telescopic chain $\mathcal{P}$ as we are now going to check.

Let $g$ be any $(a, b)$-quasi geodesic of $\hat{X}$ between $x$ and $y$. By Proposition 3.10, $g \subset N_{tel}^{C_{3.10}}(C)$. From Lemma 3.12, $\mathfrak{G} := P_{C}^{hor}(g)$ is a $D_{3.12}$-telescopic $(C_{3.12}, C_{3.12})$-quasi geodesic of $(C, d_{tel}^{D_{3.12}})$.

The quasi geodesic $\mathfrak{G}$ intersects the vertical trees $T_0, T_1, \ldots, T_r$ of $C$: let $\mathfrak{G}_0$ be the smallest subset of $g$ between $x$ and $T_0$. From Proposition 3.9, $\mathfrak{G}_0$ is contained in the $C_{3.9}$-neighborhood of the union of the vertical segments $s_0, s_1$ from the endpoints of $\mathfrak{G}_0$ to those of $h_0$. From our observation above about the exponential separation of the endpoints of $h_0$, there is some $\kappa > 0$ such that, outside the region in $C$ centered at $h_0$ with vertical width $\kappa$, the horizontal geodesics between the vertical trees of the endpoints of $h_0$ have horizontal length greater than $3C_{3.9}$. We so get a constant $K(v, L, a, b) > 0$, not depending of the quasi geodesic nor on the corridor considered, such that $d_{tel}^{H}(\mathfrak{G}_0, s_0 \cup h_0 \cup s_1) \leq K(v, L, a, b)$.

The same arguments apply for the subset $\mathfrak{G}_i$ between $T_{i-1}$ and $T_1$ until $i = r$. Since $|h_{r+1}|_{hor} \leq L$ and $h_{r+1}$ has been chosen to minimize the vertical distance between $h_r$ and all horizontal geodesics $h$ satisfying $|h|_{hor} \leq L$, we easily get a constant $K'(v, L, a, b)$ such that the concatenation of $h_{r+1}$ with

- the $v$-vertical segment in $B_1$ between $h_{r+1}$ and $y$ (the terminal point of $g$),
- the $C_{3,6}$-segment in $T_r$ between $h_r$ and $h_{r+1}$,

is at Hausdorff distance smaller than $K'(v, L, a, b)$ from the subset of $\mathfrak{G}$ following the concatenation of the $\mathfrak{G}_i$’s.

It follows that $\mathcal{P}$ is a $D_{3.12}$-telescopic chain between $x$ and $y$ with $d_{tel}^{H}(g, \mathcal{P}) \leq \max(K(v, L, a, b), K'(v, L, a, b))$. Since the construction of the horizontal geodesics $h_i$ does not depend on the endpoints $x$ and $y$, we also have the conclusion for $\mathcal{P}'$. 

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It remains to check that $\mathcal{P}$ is a $(D, D)$-quasi geodesic, with $D$ only depending on $v$ and $L$. It suffices to choose $a = 1$ and $b = 0$ and then apply what was proved just above: the chain $\mathcal{P}$ is at Hausdorff distance smaller than $\max(K(v, L, 1, 0), K'(v, L, 1, 0))$ from a geodesic. From this observation, we easily get by classical arguments and computations that $\mathcal{P}$ is a quasi geodesic as announced.  

4. APPROXIMATION OF QUASI GEODESICS: THE GENERAL CASE

In order to give a simple statement, we added in Theorem 3.8 the restriction that the attaching-maps of the tree of spaces be quasi isometries, instead of requiring that they be quasi isometric embeddings. In this way, the elementary notion of corridor (Definition 3.1) was sufficient to describe the quasi geodesics of the space. In Example 3.2, we saw that the notion of corridor was too rough to capture the geometry of the amalgamated product $\mathbb{F}_2 *_{\pi} \mathbb{F}_2$. We now define the generalized $v$-corridors that we will substitute to the corridors of Theorem 3.8 in order to obtain the more general statement we are looking for.

**Definition 4.1.** Let $(\bar{X}, \bar{T}, \pi)$ be a tree of spaces. Let $v \geq 0$. A generalized $v$-corridor $C$ in $\bar{X}$ is a union of horizontal geodesics which satisfies the following properties:

(a) For each $\alpha \in \bar{T}$, $C \cap X_\alpha$ either is empty or is a horizontal geodesic.

(b) The image $T := \pi(C)$ is a subtree of $\bar{T}$.

(c) If $w$ is a vertex of $\bar{T}$ in the above subtree $T$ and $e$ is an edge of $\bar{T}$ which is incident to $w$ but does not belong to $T$, then there is no $v$-vertical segment over $e$ starting from $C$.

(d) The subtree $T \subset \bar{T}$ admits a decomposition in subtrees $(T_i)_{i \in I}$ such that:

(a) For any $(i, j) \in I \times I$ with $i \neq j$, $T_i \cap T_j$ either is empty or is reduced to a single vertex.

(b) For each $i \in I$, there are two $v$-vertical trees $\sigma_1^i, \sigma_2^i : T_i \to \bar{X}$ such that each maximal horizontal geodesic in the subcorridor $C_i := C \cap \pi^{-1}(T_i)$ has its endpoints in $\sigma_1^i(T_i)$ and $\sigma_2^i(T_i)$.

(c) If $T_i \cap T_j$ is non-empty then $C_i \cap C_j$ is a horizontal geodesic.

With the notations above, the subset $\bigcup_{i \in I} \sigma_1^i(T_i) \cup \bigcup_{i \in I} \sigma_2^i(T_i)$ is the *vertical boundary* of the generalized $v$-corridor.

Let $h$ be a horizontal geodesic. A minimal generalized $v$-corridor through $h$ is any $v$-corridor $C(h)$ in $\bar{X}$ such that $C(h) \cap X_{\pi(h)} = h$ and such that any other $v$-corridor $C$ containing $h$ satisfies $\pi(C(h)) \subset \pi(C)$.

**Example 4.2.** We consider the tree of 0-hyperbolic spaces $(\bar{X}, \bar{T}, \pi)$ associated to the amalgamated product $\mathbb{F}_2 *_{\pi} \mathbb{F}_2$ already evoked in Examples 2.3 and 3.3. Let $h$ be a horizontal geodesic reading $x_2 x_1 x_2 x_2$ in the Cayley graph of $\mathbb{F}_2$ (a tree) over some vertex of $\bar{T}$. Figure 8 represents a minimal generalized 0-corridor $C(h)$ through $h$.

With the notations of Definition 4.1, the projection $T = \pi(C(h)) = T_1 \cup T_2 \cup T_3$ is the union of three bi-infinite geodesics $T_i$ which intersect in a unique common vertex. These are the three thick lines in Figure 8. We have $\sigma_1^1(T_1) = \sigma_2^1(T_1)$ and $\sigma_1^2(T_2) = \sigma_2^3(T_3)$. In other words each horizontal geodesics of $C_1 = \pi^{-1}(T_1) \cap C$ and $C_3 = \pi^{-1}(T_3) \cap C$ is reduced to a single point and the subcorridors $C_1$ and $C_3$ are two bi-infinite $v$-vertical segments. They pass through the endpoints of $h$. Finally the subcorridor $C_2 = \pi^{-1}(T_2) \cap C$ is a minimal generalized 0-corridor containing the horizontal geodesic $x_1^2$, see Example 3.3.
Figure 8.

Observe that the notions of 0-corridors and generalized 0-corridors are very different since we saw in Example 3.3 that the unique 0-corridor containing \( h = x_2 x_1^2 x_2 \) is \( h \).

**Example 4.3.** With the notations of Example 4.2, let \( x \) be an endpoint of the horizontal geodesic \( h \). Let \( y \) be any other vertex in \( \tilde{X} \) such that \( \pi(y) \notin T \). Then of course \( y \notin \mathcal{C}(h) \).

Consider the unique shortest \( T \)-geodesic from \( \pi(y) \) to \( T \). This is an edge-path \( e_1 \cdots e_k \). For simplicity, assume there is a single edge \( e \). Then there are 0-vertical segments over \( e \) from some bi-infinite geodesic \( \mathcal{A} \) reading \( x_1^\pm \infty \) in \( X_{\pi(e)} = X_{\pi(y)} \) to another bi-infinite geodesic \( \mathcal{A}' \) reading \( x_1^\pm \infty \) in \( X_{t(e)} \) \( (t(e) \in T) \). Let \( h_0 \) (resp. \( h_1 \)) be the shortest horizontal (i.e. \( \mathbb{F}_2 \)-) geodesic in \( X_{\pi(e)} \) (resp. in \( X_{t(e)} \)) from \( y \) (resp. from \( \mathcal{C}(h) \cap X_{t(e)} \)) to \( \mathcal{A} \) (resp. to \( \mathcal{A}' \)). Then the union \( \mathcal{C}(h) \cup \mathcal{C}(h_0) \cup \mathcal{C}(h_1) \) is a generalized 0-corridor the vertical boundaries of which pass through \( x \) and \( y \). Beware however that, since the projections to \( T \) of the \( \mathcal{C}(h_i) \)'s overlap, they do not form the subcorridors of Definition 4.1. One has to further decompose to get the subcorridors. Figure 9 illustrates what a “typical” generalized corridor looks like.

Figure 9.

A slight generalization of the construction above allows one to get the following statement:
Lemma 4.4. Let $\tilde{X}$ be a tree of hyperbolic spaces. There exist $D \geq C_{2,16}$ such that, for any $v \geq D$ and for any two distinct points $x, y$ in $\tilde{X}$ there is a generalized $v$-corridor the vertical boundaries of which pass through $x$ and $y$.

Theorem 4.5. Let $\tilde{X}$ be a tree of hyperbolic spaces which satisfies the hallways-flare property. Then:

- for any $v \geq C_{2,16}$, there is $E \geq v$,
- for any $L > 0$ greater than some critical constant and for any $v \geq C_{2,16}$ there exists $D \geq 1$,
- for any $L > 0$ greater than some critical constant, for any $v \geq C_{2,16}$, for any $a \geq 1$ and $b \geq 0$ there exists $C \geq 0$,

such that for any pair of distinct points $x, y$ in $\tilde{X}$, for any generalized $v$-corridor $C$ whose vertical boundaries pass through $x$ and $y$, there is a $E$-telescopic chain $P$ in $\mathcal{C}$ satisfying the following properties:

(a) This is a $(D, D)$-quasi geodesic.
(b) At the exception of at most one in each subcorridor (see Definition 4.1), each maximal horizontal path in $P$ is a diagonal with horizontal length greater or equal to $L$.
(c) For any $(a, b)$-quasi geodesic $g$ in $\tilde{X}$ with endpoints $x$ and $y$, the Hausdorff distance between $g$ and $P$ is bounded above by $C$.
(d) Let $P'$ be the closed complement in $P$ of the first and last maximal vertical segments. Let $g'$ be any $(a, b)$-quasi geodesic in $\tilde{X}$ whose endpoints lie in the vertical boundaries of $C$. Let $s_0, s_1$ be the $v$-vertical segments in the vertical boundaries of $C$ from the endpoints of $P'$ to the endpoints of $g'$. Then $g'$ is at Hausdorff distance smaller than $C$ from the concatenation of $P'$ with $s_0$ and $s_1$.

Proof of Theorem 4.5: We first need an adaptation to this more general setting of some of the lemmas and propositions given for proving Theorem 3.8:

Proposition 4.6. Lemma 3.6, Proposition 3.9 and Proposition 3.10 remain true for generalized $v$-corridors with $v \geq D_{4,4}$.

There is nothing to prove with respect to Lemma 3.6 and Proposition 3.9. We refer the reader to Section 8.6 for the proof of the adaptation of Proposition 3.10 to generalized corridors.

Lemma 4.7. Let $\tilde{X}$ be a tree of hyperbolic spaces. For any $v \geq D_{4,4}$, for any $a \geq 1$ and $b \geq 0$ there exists $C \geq 0$ such that if $g$ is any $(a, b)$-quasi geodesic, if $\mathcal{C}$ is any generalized $v$-corridor the vertical boundaries of which pass through the endpoints of $g$, then there is a $(a, b + 2\delta)$-quasi geodesic $G$ with $d_{\tilde{X}}^H(g, G) \leq C$ and $\pi(G) \subset \pi(\mathcal{C})$.

Proof: Let $\gamma \in \mathcal{T}$ be an endpoint of $\pi(\mathcal{C})$. Assume that $g'$ is a maximal subset of $g$ with endpoints in $X_\gamma$ and such that $\pi(g') \cap \pi(\mathcal{C}) = \gamma$. Then, since $v \geq C_{2,16}$, Lemma 2.16 tells us that the endpoints of $g'$ are $2\delta$-close with respect to the horizontal distance. Since $g$ is a $(a, b)$-quasi geodesic, $g'$ is $(2a\delta + b)$-close to $X_\gamma$ with respect to the telescopic distance. Substituting $g'$ by a horizontal geodesic between its endpoints and repeating this substitution for all the subsets of $g$ like $g'$ yields a quasi geodesic as announced. □

With the above adaptations in mind, the proof of Theorem 4.5 is now a duplicate of the proof of Theorem 3.8. □
5. Hyperbolicity and Weak Relative hyperbolicity

5.1. Hyperbolicity of trees of spaces. Theorem 5.1 generalizes Bestvina-Feighn’s combination to non-proper hyperbolic spaces. Bowditch proposed such a generalization in [6].

Theorem 5.1. Let $\tilde{X}$ be a tree of hyperbolic spaces which satisfies the hallways-flare property. Then $\tilde{X}$ is a Gromov-hyperbolic metric space.

Proof of Theorem 5.1: We begin by proving the

Theorem 5.2. Let $\tilde{X}$ be a tree of hyperbolic spaces which satisfies the hallways-flare property. For any $a \geq 1$ and $b \geq 0$ there exists $C \geq 0$ such that $(a, b)$-quasi geodesic bigons are $C$-thin.

Proof: By Lemma 2.13, it suffices to prove Theorem 5.2 for $D_{4,4}$-telescopic $(a, b)$-quasi geodesic bigons. Thus let $g_0, g_1$ be the two sides of a $D_{4,4}$-telescopic $(a, b)$-quasi geodesic bigon. By Theorem 4.5 (Theorem 3.8 suffices in the case where the attaching-maps of $\tilde{X}$ are quasi isometries), there is $E_{4,5} \geq D_{4,4}$ and a $E_{4,5}$-telescopic chain $P$ such that for $i = 0, 1$ we have $d^H(g_i, P) \leq C_{4,5}$. Hence $d^H(g_0, g_1) \leq 2C_{4,5}$ and Theorem 5.2 is proved. □

The following lemma was first indicated to the author by I. Kapovich:

Lemma 5.3. [13] Let $(X, d)$ be a $(r, s)$-quasi geodesic space. If for any $r' \geq r, s' \geq s$, there exists $\delta(r', s')$, such that $(r', s')$-quasi geodesic bigons are $\delta(r', s')$-thin, then $(X, d)$ is a $2\delta(r, 3s)$-hyperbolic space.

Theorem 5.2 together with Lemma 5.3 imply Theorem 5.1. □

5.2. Weak relative hyperbolicity.

Theorem 5.4. Let $(G, \mathcal{H}_v, \mathcal{H}_e)$ be a graph of weakly relatively hyperbolic groups. If some relative $(G, \mathcal{H}_v, \mathcal{H}_e)$-tree of spaces satisfies the hallways-flare property, then the fundamental group of $G$ is weakly hyperbolic relative to the family formed by all the parabolic subgroups of the vertex-groups.

Proof: By Theorem 5.1, this readily follows from the definitions. □

Proof of Theorem 1.5, weak relative hyperbolicity case. With a slight abuse, we consider $\mathbb{F}_r$ as a subgroup of $\text{Aut}(G, \mathcal{H})$ generated by the automorphisms $\alpha_i$’s. The group $G \rtimes \mathbb{F}_r$ is the fundamental group of the graph of groups which has $G$ as unique vertex group $G_v$ and $G$ as the $r$ edge-groups $G_{e_i}$ (the $e_i$’s are loops with initial and terminal vertex $v$). The attaching endomorphisms $\iota_{e_i}: G_{e_i} \hookrightarrow G_v$ are the automorphisms $\alpha_i$ whereas the $\iota_{e_i}^{-1}: G_v \hookrightarrow G_{e_i}$ are the identity. Since the $\alpha_i$’s are relative automorphisms of $(G, \mathcal{H})$, each one induces a quasi isometry from $(G_{e_i}, \mathcal{H})$ to $(G_v, \mathcal{H})$. Since $\mathbb{F}_r$ is a uniform free group of relatively hyperbolic automorphisms, any relative $(G, \mathcal{H}_v, \mathcal{H}_e)$-tree of spaces satisfies the hallways-flare property. The weak relative hyperbolicity case of Theorem 1.5 is then a corollary of Theorem 5.4. □

From [19], a hyperbolic group is weakly hyperbolic relative to any finite family of quasi convex subgroups. We so get:

Corollary 5.5. Let $G$ be a hyperbolic group, let $\mathcal{H}$ be a finite family of infinite subgroups of $G$ and let $\alpha \in \text{Aut}(G, \mathcal{H})$ be hyperbolic relative to $\mathcal{H}$. If $\mathcal{H}$ is quasi convex in $G$ then the mapping-torus group $G_\alpha = G \rtimes_\alpha \mathbb{Z}$ is weakly hyperbolic relative to $\mathcal{H}$.
Proof of Theorem 1.9: We are going to prove that if the hallways-flare property is not satisfied then the property given in Theorem 1.9 neither is. Theorem 5.4 then gives the conclusion.

If the hallways-flare property is not satisfied then by Lemma 2.16, for any $v \geq C_{2,16}$, for any $\lambda > 1$, $M, t_0 > 0$ there exists $\alpha, \beta, \gamma \in \mathcal{T}$ with $\alpha \in [\beta, \gamma]$, $d_T(\beta, \alpha) = d_T(\gamma, \alpha) = t_0$ and a pair of $v$-vertical segments $s_0, s_1$ over $[\beta, \gamma]$ with $d_{hor}(s_0 \cap X_\alpha, s_1 \cap X_\alpha) \geq M$ satisfying $d_{hor}(s_0 \cap X_\beta, s_1 \cap X_\beta) < \lambda d_{hor}(s_0 \cap X_\alpha, s_1 \cap X_\alpha)$ and $d_{hor}(s_0 \cap X_\gamma, s_1 \cap X_\gamma) < \lambda d_{hor}(s_0 \cap X_\alpha, s_1 \cap X_\alpha)$. By choosing $v$ sufficiently large enough with respect to the constants $\delta, a$ and $b$ of Definition 2, and then $M$ sufficiently enough with respect to $v$ we get a sequence of four points $x_n, y_n, z_n, t_n$ such that $[z_n, t_n]$ lies in the 28-neighborhood in $X_\alpha$ of $[x_n, y_n]$ and the sequence of the horizontal geodesics $[x_n, y_n], [z_n, t_n]$ contradicts the property given in Theorem 1.9.

\[ \square \]

6. Strong relative hyperbolicity

Let $(\hat{X}_S, d_S)$ be a coned space (see the definition of weak relative hyperbolicity in the Introduction) and for $u \geq 1, v \geq 0$ let $\hat{g}$ be a $(u, v)$-quasi geodesic path in $(\hat{X}_S, d_S)$. A trace $g$ of $\hat{g}$ in $(X, d)$ is obtained by substituting each subpath of $\hat{g}$ not in $(X, d)$ by a subpath of $(X, d)$ in $S$ with same endpoints, which is a geodesic for the metric induced by $X$ on $S$. We say that $g$ (or $\hat{g}$) backtrack if $g$ reenters a subset in $S$ that it left before.

Definition 6.1. [11] A coned space $(\hat{X}_S, d_S)$ satisfies the Bounded-Coset Penetration property (BCP) (with respect to $S$) if and only if for any $u \geq 1$ and $v \geq 0$ there exists $C \geq 0$ such that, for any two $(u, v)$-quasi geodesics $\hat{g}_0, \hat{g}_1$ of $(\hat{X}_S, d_S)$ with traces $g_0, g_1$ in $(X, d)$, which have the same initial point, which have terminal points at most $1$-apart in $(X, d)$ and which do not backtrack, the following two properties are satisfied:

(a) if both $g_0$ and $g_1$ intersects a set $S_i \in S$ then their first intersection points with $S_i$ are $C$-close in $(X, d)$,

(b) if $g_0$ intersects a set $S_i$ that $g_1$ does not, then the length in $(X, d)$ of $g_0 \cap S_i$ is smaller than $C$.

Definition 6.2. [11] A quasi geodesic space $(X, d)$ is strongly hyperbolic relative to a family of subsets $S$ if and only if the $S$-coned space $(\hat{X}_S, d_S)$ is Gromov hyperbolic and satisfies the BCP.

Let $G$ be a group with finite generating set $S$ and associated Cayley graph $\Gamma_S(G)$, and let $\mathcal{H} = \{H_1, \ldots, \}$ be a family of infinite subgroups $H_i$ of $G$.

The group $G$ is strongly hyperbolic relative to $\mathcal{H}$ if and only if $\Gamma_S(G)$ is strongly hyperbolic relative to the union of the left-classes $xH_i$, i.e. if and only if $\Gamma_{\mathcal{H}}(G)$ is hyperbolic and satisfies the BCP.

Definition 6.3. Let $(G, \mathcal{H}_v, \mathcal{H}_e)$ be a fine graph of strongly relatively hyperbolic groups. Let $(\hat{X}, \mathcal{T}, \pi)$ be some relative $(G, \mathcal{H}_v, \mathcal{H}_e)$-tree of spaces.

An exceptional orbit in $\hat{X}$ is a maximal set $S$ of exceptional vertices in $\hat{X}$ such that $(v, v') \in S \times S$ if and only if $\hat{j}_{\pi(v), \pi(v')}(v) = v'$.

In the strong relative hyperbolicity setting, the parabolic subgroups are necessarily almost malnormal (see definition in Section 6.1). Since the parabolic subgroups are assumed to be infinite, this readily implies that there is no choice for defining the image of an exceptional vertex under $\hat{j}$. Thus exceptional orbits are uniquely defined by the structure of $G$ as a graph of relatively hyperbolic groups.
Definition 6.4. Let \((\mathcal{G}, \mathcal{H}_v, \mathcal{H}_e)\) be a fine graph of strongly relatively hyperbolic groups. A relative \((\mathcal{G}, \mathcal{H}_v, \mathcal{H}_e)\)-tree of spaces \((\hat{X}, T, \pi)\) satisfies the strong hallways-flare property if it satisfies the hallways-flare property if and only if, for any \(x \geq 0\), there is \(t \geq 0\) such that the vertical width of any region where two exceptional orbits remain at horizontal distance smaller than \(m\) one from each other is smaller than \(t\).

Theorem 6.5. Let \(\mathcal{G}\) be a fine graph of strongly relatively hyperbolic groups. If some relative \((\mathcal{G}, \mathcal{H}_v, \mathcal{H}_e)\)-tree of spaces satisfies the strong hallways-flare property, then the fundamental group of \(\mathcal{G}\) is strongly hyperbolic relative to a family formed

- by exactly one representative from each finite parabolic orbit,
- by the free extensions of exactly one representative from each infinite parabolic orbit.

We begin the proof with the

Lemma 6.6. With the assumptions and notations of Theorem 6.5: There exists \(C \geq 0\) such that any exceptional orbit is a discrete subset of a \(C\)-vertical tree.

Proof: Since there are only finitely many parabolic subgroups preserved up to conjugacy and since the free groups which fix these subgroups up to conjugacy are finitely generated, there are only finitely many conjugation elements. Let \(m\) be the maximum of their word-lengths. Then \(m + \frac{1}{2} (\frac{1}{2} \text{ for going from an exceptional vertex of } \hat{X} \text{ to } X \text{ plus } 1 \text{ for going through a left } H\text{-class coned in } \hat{X})\) gives the announced constant. □

The following lemma is a straightforward consequence of the strong hallways-flare property:

Lemma 6.7. With the assumptions and notations of Theorem 6.5: There exists \(C \geq 0\) such that, if \(v, v'\) are any two exceptional vertices in a same stratum of \(\hat{X}\) then the exceptional orbits of \(v\) and \(v'\) are connected by a diagonal (see Definition 3.7) of horizontal length greater or equal to \(1\), the endpoints of which are exponentially separated in all the directions outside a region whose vertical width is smaller than \(C\).

Lemma 6.8. With the assumptions and notations of Theorem 6.5: For any \(a \geq 1\) and \(b \geq 0\) there exists \(C \geq 0\) such that if \(g, g'\) are two \((a, b)\)-quasi geodesics of \(\hat{X}\) between two exceptional orbits \(L_1, L_2\) then \(g, g'\) admit decompositions \(g = g_1g_2g_3\) and \(g' = g'_1g'_2g'_3\) with the following properties: \(g_1 \in N_X^C(L_1), g'_1 \in N_X^C(L_1), g_3 \in N_X^C(L_2), g'_3 \in N_X^C(L_2)\) and \(d_X^H(g_2, g'_2) \leq C\). If \(g\) and \(g'\) have the same endpoints then \(d_X^H(g, g') \leq C\).

Proof: This is an easy consequence of Theorem 4.5. For simplicity assume that the attaching-maps of \(\hat{X}\) are quasi isometries so that Theorem 3.8 can be applied. The \(C_{6,6}\)-vertical trees through the given two exceptional orbits bound a \(C_{6,6}\)-corridor. Both \(g\) and \(g'\) are approximated by two chains \(G\) and \(G'\) which only possibly differ by their first and last maximal vertical segments in \(L_1\) and \(L_2\). These last vertical segments are where \(g\) and \(g'\) are not necessarily close one to each other if they don’t have the same endpoints but are close to the given exceptional orbits. As written before, the extension to the general case where there is not a corridor, but only a generalized corridor, between the two exceptional orbits, is easily dealt with by using Theorem 4.5 instead of Theorem 3.8. □
**Definition 6.9.** Let $(\mathcal{G}, \mathcal{H}_v, \mathcal{H}_e)$ be a fine graph of strongly relatively hyperbolic groups. An *exceptional space* $C(\hat{X})$ is the metric space obtained from a relative $(\mathcal{G}, \mathcal{H}_v, \mathcal{H}_e)$-tree of spaces $\hat{X}$ by putting a cone over each exceptional orbit.

Lemma 6.10 below stresses the importance of this exceptional space.

**Lemma 6.10.** With the assumptions and notations of Theorem 6.5: The exceptional space $C(\hat{X})$ is hyperbolic and satisfies the BCP with respect to the exceptional orbits if and only if the fundamental group of $\mathcal{G}$ is strongly hyperbolic relative to a family of subgroups as given by Theorem 6.5.

**Remark 6.11.** Let $G$ be a finitely generated group which is strongly hyperbolic relative to $\mathcal{H} = \{H_1, H_2\}$. Let $\alpha \in \text{Aut}(G, \mathcal{H})$ such that $\alpha(H_1)$ is a conjugate of $H_2$ and $\alpha(H_2)$ is a conjugate of $H_1$. Consider the graph of groups $\mathcal{G}$ with one vertex and one edge (a loop) associated to $G \rtimes_{\alpha} \mathbb{Z}$. Let $\hat{X}$ be a relative $(\mathcal{G}, \mathcal{H}, \mathcal{H})$-tree of spaces. Then in $C(\hat{X})$ cones are put above the left $H_i$-classes, and their exceptional vertices all belong to a same exceptional orbit. However only one of the two subgroups $H_1, H_2$ appears in the subgroups of the relative part described by Theorem 6.5 because otherwise the condition of almost malnormality would be violated.

**Proof of Lemma 6.10:** Let $\mathcal{Y}$ be the space obtained by coning $\hat{X}$ according to the parabolic subgroups described in Theorem 6.5. The essential difference between $\mathcal{Y}$ and the coned space $C(\hat{X})$ of Definition 6.9 is the following one:

In $C(\hat{X})$ a horizontal cone is first put over all the left-classes for the parabolic subgroups in the edge and vertex groups; then a “vertical cone” is put over all the vertices which belong to a same exceptional orbit. In $\mathcal{Y}$, a cone is put on the left-classes of exactly one subgroup from each finite orbit, and of exactly one free extension of subgroup in each infinite orbit.

Observe that in both $C(\hat{X})$ and $\mathcal{Y}$, there is exactly one exceptional vertex for each exceptional orbit. One thus has a natural one-to-one correspondence, denoted by $B$, between the exceptional vertices of $C(\hat{X})$ and those of $\mathcal{Y}$. Assume that there is a horizontal cone in $C(\hat{X})$ over two exceptional vertices $x, y$ in a same stratum of $\hat{X}$. It belongs to an exceptional orbit and we denote by $v(gH)$ the exceptional vertex associated to this leaf. Consider the exceptional vertex $B(v(gH))$ of $\mathcal{Y}$. Assume that $x, y$ do not belong to the cone with vertex $B(v(gH))$. Then there are two points $x', y'$ in another stratum which are at bounded telescopic distance from $x$ and $y$ and belong to this cone. This is straightforward if $v(gH)$ is the vertex of the cone over a finite exceptional orbit. Otherwise this comes from the finite generation of the free groups which preserve the parabolic subgroups up to conjugacy and from the fact that there is an upper-bound on the length of the conjugacy elements.

There is a natural map $j: \mathcal{Y} \rightarrow C(\hat{X})$ whose restriction to $\hat{X}$ is the identity-map and which maps each exceptional vertex $v(gH)$ of $\mathcal{Y}$ to the exceptional vertex $B^{-1}(v(gH))$ of $C(\hat{X})$. The observation of the previous paragraph readily implies the following assertion: if $g$ is a quasi geodesic of $\mathcal{Y}$, then $j(g)$ is a quasi geodesic of $C(\hat{X})$ (with possible different constants of quasi geodesicity) and traces in $\hat{X}$ of $g$ and $j(g)$ are Hausdorff-close. The lemma follows. \(\square\)

**Remark 6.12.** The hyperbolicity of the coned space $C(\hat{X})$ follows from the quasi convexity of the exceptional orbits implied by Lemma 6.6 and from the arguments developed for proving Proposition 1 of [30]. However we re-prove it when listing below the arguments for checking the BCP.
Lemma 6.13. With the assumptions and notations of Theorem 6.5: For any \( v \geq D_{4,4}, \) for any \( a \geq 1 \) and \( b, r \geq 0 \) there exists \( C \geq 0 \) such that if \( g_1, g_2 \) are two \((a, b)\)-quasi geodesics of \( C(\mathring{X}) \), the terminal points of which are at most 1-apart in \( \mathring{X} \), and with same initial point in \( \mathring{X} \), if \( C \) is a generalized \( v \)-corridor whose vertical boundaries pass through the endpoints of \( g_1 \), if traces \( \widehat{g}_i \)'s of the \( g_i \)'s in \( \mathring{X} \) satisfy \( \widehat{g}_i \subset N^r_C(C) \) for \( i = 1, 2 \) then \( d^H_{C(\mathring{X}))(g_1, g_2)} \leq C \). Furthermore, if \( g_1 \) and \( g_2 \) do not backtrack then they satisfy the two conditions required by the BCP with a constant \( D \) depending on \( v, a, b, r \).

We emphasize that this proposition is false if one only requires a bound on the distance in \( C(\mathring{X}) \) from the \( g_i \)'s to \( C \).

**Proof:** For simplicity we assume that \( C \) is a corridor, the adaptation to generalized corridors is straightforward. We consider the horizontal quasi projections on \( C \) of the maximal subsets of \( g_1, g_2 \) which belong to \( \mathring{X} \). From Lemma 3.12, these projections are \((C_{3,12}, C_{3,12})\)-quasi geodesics. From Lemmas 6.6, 6.7 on the one hand and Lemma 2.18 on the other hand, there is \( K \), depending on \( r \) and \( C_{3,6}(C_{6,6}) \), such that the horizontal quasi projections of the exceptional orbits are \( K \)-vertical trees, for which there exists a constant \( L \) playing the rôle of the constant \( t_{6,4} \). It is equivalent to prove the announced properties for the bigon \( g_1, g_2 \) with respect to the exceptional orbits than to prove them for the above projections on \( C \).

If \( g_1, g_2 \) go through the same exceptional orbits, then their horizontal quasi projections on \( C \) satisfy the same property with respect to the horizontal quasi projections of the exceptional orbits. From Lemma 6.8, the “bigon” obtained by projection to the generalized corridor is thin. Moreover the points where the projections of \( g_1 \) and \( g_2 \) penetrate a given exceptional orbit are close, because either they are close to the diagonal preceding this exceptional orbit, or they leave a same exceptional orbit: in this last case we are done by the existence of the constant \( L \) above (the analog on the corridor of the constant \( t_{6,4} \)). Let us now assume that \( g_1 \) enters in an exceptional leaf \( S \) but \( g_2 \) does not. Of course this also holds for the respective projections on \( C \). We then distinguish three cases:

*First case: the exit point of \( g_1 \) is followed by a diagonal with horizontal length greater than some constant \( D \) (depending on the constants of hyperbolicity and exponential separation).* Then (the projection of) \( g_2 \) has to go to a bounded neighborhood of this diagonal, this is Theorem 3.8. It remains before in a bounded horizontal neighborhood of the exceptional orbit, the bound depending on \( a, b \) and \( r \) (since the constants of quasigeodesicity of the projections depend on \( r \)). Thus the vertical length of the passage of \( g_1 \) through this exceptional orbit is bounded above by a constant depending on \( a, b \) and \( r \).

*Second case: the exit point of \( g_1 \) is followed by another exceptional orbit.* Thanks to the existence of the constant \( L \) and Lemma 6.7, we can follow the same arguments as above, appealing to Proposition 3.9 rather than directly Theorem 3.8. We leave the reader work out details and computations.

*Third case: the exit point of \( g_1 \) is followed by a horizontal geodesic with horizontal length bounded above by the constant of the first case.* In this case, this horizontal geodesic ends at the vertical boundary of \( C \). The entrance-point of \( g_1 \) in \( S \) is close to a point in \( g_2 \). Since \( g_2 \) is a \((a, b)\)-quasi geodesic and \( g_2 \) does not pass through \( S \), it cannot happen that the passage of \( g_1 \) though \( S \) is a long passage at small horizontal distance from the considered vertical boundary. Thus, if it is a long passage then there is a stratum which is closest to the entrance-point of \( g_1 \) in \( S \) and where the horizontal distance between \( S \) and the considered vertical boundary is smaller than the critical constant. From Proposition 3.9,
Theorem 6.5. \[ \text{Let } g_2 \text{ lies in a bounded neighborhood of } S \text{ until reaching this stratum. Once again, this gives an upper-bound on the vertical length of } S. \]

The proof of Lemma 6.13 now follows in an easy way: to conclude for the BCP, we need of course the fact that the horizontal metrics on the strata satisfy the BCP. \[
\]

**Proposition 6.14.** With the assumptions of Lemma 6.13: For any \( v \geq D_{a,4} \), for any \( a \geq 1 \) and \( b \geq 0 \) there exist \( C \geq 1 \) and \( D > 0 \) such that, if \( x_0, x_1, \ldots, x_n \) are consecutive points in some exceptional orbit \( L \), which lie outside the horizontal \( D \)-neighborhood of a generalized \( v \)-corridor \( C \), and if the vertical distance between the strata of \( x_0 \) and \( x_n \) is greater than \( C \), then no non-backtracking \((a, b)\)-quasi geodesic of \( C(\hat{X}) \) with both endpoints in the horizontal \( D \)-neighborhood of \( C \) contains the cone over \( \{x_0, x_n\} \).

See proof in subsection 8.7.

**Proof of Theorem 6.5:** Let \( g, g' \) be two non-backtracking \((a, b)\)-quasi geodesics of \( C(\hat{X}) \) with same initial point, and with terminal points at most \( 1 \)-apart in \( \hat{X} \). We assume for simplicity that the attaching-maps of \( \hat{X} \) are quasi isometries, the adaptation to the general case is easy. There is a corridor \( C \) (in the whole generality only a generalized corridor) the vertical boundaries of which pass through the initial and terminal points of \( g \).

Let \( p \) be a passage of \( g \) (resp. \( g' \)) through the cone over a subset \( S \) of an exceptional orbit outside the \( D_{6,14} \)-neighborhood of \( C \) in \( \hat{X} \). From Proposition 6.14, substituting \( p \) by \( S \) yields a non-backtracking \((\kappa(a, b), \kappa'(a, b))\)-quasi geodesics \( h \) (resp. \( h' \)) of \( C(\hat{X}) \), with \( \kappa(a, b) = C_{6,14} * C_{6,6} * a \) and \( \kappa'(a, b) = C_{6,14} * C_{6,6} * (b+1) \), such that \( d^{H}_{C(\hat{X})}(g, h) \leq 1 \) (resp. \( d^{H}_{C(\hat{X})}(g', h') \leq 1 \)). We can thus assume that all passages like \( p \) have been suppressed in \( h \) and \( h' \) as above.

By Proposition 3.10, the subsets of \( h \) and \( h' \) between two exceptional orbits are contained in the horizontal \( C_{3,10} \)-neighborhood of a corridor between these orbits. Thus \( h \) and \( h' \) are contained in the \( D_{6,14} + C_{3,10} \)-neighborhood of \( C \) in \( \hat{X} \). From Lemma 6.13, \( h, h' \) satisfy the BCP. The conclusion for \( g, g' \) follows.

The proof of the hyperbolicity follows the same scheme. If \( g, g' \) form a \((a, b)\)-quasi geodesic bigon of \( C(\hat{X}) \), one first substitutes it by a non-backtracking \((a, b)\)-quasi geodesic bigon \( \hat{g}_0, \hat{g}_0' \) with \( d^{H}_{C(\hat{X})}(g, \hat{g}_0) \leq b \), \( d^{H}_{C(\hat{X})}(g', \hat{g}_0') \leq b \). The line of the arguments thereafter is the same as above: at the end, Lemma 6.13 gives the thinness of the quasi geodesic bigons instead of the BCP. As in Section 5.1, the hyperbolicity follows from Lemma 5.3. \[
\]

### 6.1. Some corollaries of Theorem 6.5.

**Definition 6.15.** Let \( G \) be a group. A finite family \( \{H_1, \cdots, H_k\} \) of subgroups of \( G \) is almost malnormal if and only if:

1. for any \( i = 1, \cdots, k \), \( H_i \) is almost malnormal in \( G \), i.e. the cardinality of the set \( \{w \in H_i \mid \exists g \notin H_i \text{ s.t. } w \in g^{-1}H_ig\} \) is finite.

2. for any \( i, j \in \{1, \cdots, k\} \) with \( i \neq j \), the cardinality of the set \( \{w \in H_j \mid \exists g \in G \text{ s.t. } w \in g^{-1}H_ig\} \) is finite.

If the family of subgroups consists of only one subgroup, the definition above is nothing else than the definition of almost malnormality of this subgroup. It is known (see [11]) that if a group \( G \) is strongly hyperbolic relative to a family \( \mathcal{H} \) then \( \mathcal{H} \) is almost malnormal in \( G \). Otherwise the BCP is violated.
Proof of Theorem 1.5, strong relative hyperbolicity case. It suffices to check that the definition of a uniform free group of relatively hyperbolic automorphisms implies the strong hallways-flare property. The exponential separation of the vertical segments is clear but one has to prove that any two exceptional orbits also separate exponentially one from each other. Assume that this is not satisfied. Then, there is $M \geq 1$, $\alpha \in \text{Aut}(G, \mathcal{H})$, and $\beta \in \text{Aut}(G, \mathcal{H})$, $|\alpha \beta \alpha^{-1} \beta^{-1}|_{\mathcal{H}} \leq M$.

Here $H$ and $H'$ are the parabolic subgroups of $G$ corresponding to the left-classes associated to the two exceptional orbits which violate, for the considered $\alpha$, the strong exponential separation property. The existence of $u$ above comes from the finiteness of the family $\mathcal{H}$ and from the finite generation of $G$: they imply together that there are only finitely many geodesic words of a given form which have relative length smaller than $M$.

Since $G$ is strongly hyperbolic relative to $\mathcal{H}$, $\mathcal{H}$ is almost malnormal in $G$. This readily implies, by choosing elements in $H$ and $H'$ which are sufficiently long enough in $(G, |.|_S)$, that there is an element $g$ of the form $H u H'$. $H u^{-1} H' = H u H' u^{-1} H$ which is not conjugate to an element of a parabolic subgroup. Furthermore $g$ can be chosen not to be a torsion element. From Corollary 4.20 of [27], \( \lim_{n \to +\infty} |g^n|_{\mathcal{H}} = +\infty \). However $\alpha_w(g)$ has the form $r H u H' s^{-1} H' u^{-1} H r^{-1}$. Thus $|\alpha_w(g^n)|_{\mathcal{H}} \leq |g^n|_{\mathcal{H}} + 2 |r|_{\mathcal{H}}$. Since $|r|_{\mathcal{H}}$ is a constant only depending on $|w|_{\mathcal{H}}$, by choosing $n$ sufficiently large enough we get a contradiction with the uniform hyperbolicity of $\mathbb{F}_r$.

It is now widely known that a hyperbolic group $G$ is strongly hyperbolic relative to any almost malnormal finite family of quasi convex subgroups. As a corollary of the previous theorem we thus have:

**Corollary 6.16.** Let $G$ be a hyperbolic group, let $\mathcal{H}$ be a finite family of infinite subgroups of $G$ and let $\alpha \in \text{Aut}(G, \mathcal{H})$ be hyperbolic relative to $\mathcal{H}$. If $\mathcal{H}$ is quasi convex and almost malnormal in $G$ then the mapping-torus group $G_\alpha = G \rtimes_\alpha \mathbb{Z}$ is strongly hyperbolic relative to the mapping-torus of $\mathcal{H}$.

This corollary may be specialized to torsion free one-ended hyperbolic groups, and so in particular to fundamental groups of surfaces. We so re-prove the result of [16]. Since there we gave only an idea for the statement and the proof in the Gromov relative hyperbolicity case, we include here the full statement of this result:

**Corollary 6.17.** Let $G$ be a torsion free one-ended hyperbolic group and let $\alpha$ be an automorphism of $G$. Let $\mathcal{H}$ be a family of maximal subgroups of $G$ which consist entirely of elements on which $\alpha$ acts up to conjugacy periodically or with linear growth and such that each element on which $\alpha$ acts up to conjugacy periodically or with linear growth is conjugate to an element in a subgroup in $\mathcal{H}$. Then $G_\alpha$ is weakly hyperbolic relative to $\mathcal{H}$, and strongly hyperbolic relative to the mapping-torus of $\mathcal{H}$.

If $G$ is the fundamental group of a compact surface $S$ (possibly with boundary) with negative Euler characteristic and if $h$ is a homeomorphism of $S$ inducing $\alpha$ on $\pi_1(S)$ (up to inner automorphism), then the subgroups in $\mathcal{H}$ are:
(i) the cyclic subgroups generated by the boundary curves,
(ii) the subgroups associated to the maximal subsurfaces which are unions of components on which $h$ acts periodically, pasted together along reduction curves of the Nielsen-Thurston decomposition,
(iii) the cyclic subgroups generated by the reduction curves not contained in the previous subsurfaces.

Proof: From Corollary 6.16, we only have to prove that the considered automorphism $\alpha$ of $G$ is hyperbolic relative to the given family of subgroups. The passage from the surface case to the torsion free one-ended hyperbolic group case is done thanks to the JSJ-decomposition theorems of [4]. We refer the reader to [16] for more precisions and concentrate on the surface case. The fundamental group of $S$ is the fundamental group of a graph of groups $G$ such that:

- the edge groups are cyclic subgroups associated to the reduction curves and boundary components,
- the vertex groups are the subgroups associated to the pseudo-Anosov components (type I vertices) and to the maximal subsurfaces with no pseudo-Anosov components (type II vertices),
- the (outer) automorphism $\alpha$ induced by the homeomorphism preserves the graph of groups structure.

We consider the universal covering of $G$ and the associated tree of spaces. We measure the length of a geodesic in this tree of spaces as follows:

- we count zero for the passages through the edge-spaces and through the type II vertex-spaces,
- we measure the length of the pieces through the type I vertex-spaces by integrating against the stable and unstable measures of the invariant foliations (a boundary-component is considered to belong to both invariant foliations and so the contribution of a path in such a leaf amounts to zero).

There is $N \geq 1$ such that, when the total stable (resp. unstable) length of a geodesic in a type I-vertex space is two times its unstable (resp. stable) length, then it is dilated by a factor $\lambda > 1$ under $N$ iterations of $\alpha^{-1}$ (resp. of $\alpha$). In the other cases, we find $N \geq 1$ such that the total length is dilated under $N$ iterations of both $\alpha$ and $\alpha^{-1}$. Similar computations have been presented in [16]. The conclusion of the relative hyperbolicity of $\alpha$ now comes easily since pieces with positive length, dilated either under $\alpha^N$ or under $\alpha^{-N}$, and pieces with zero length alternate. $$\square$$

A semi-direct product is only a particular case of HNN-extension. Alibegovic in [1], Dahmani in [10] or Osin in [29] treat acylindrical HNN-extensions and amalgated products. Let us now give a theorem about non-acylindrical HNN-extensions. Theorem 6.20 below deals with injective, not necessarily surjective, endomorphisms of relatively hyperbolic groups.

**Definition 6.18.** Let $G$ be a group and let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a finite family of subgroups of $G$. A subgroup $H'$ of $G$ is **almost malnormal relative to $\mathcal{H}$** if and only if there is an upper-bound on the $\mathcal{H}$-word length of the elements in the set $\{w \in H' \ ; \ \exists g \in G - H' \text{ with } w \in g^{-1}H'g\}$.

If $\mathcal{H}$ is empty in the definition above, we get the usual notion of almost malnormality of a subgroup. If in addition there is no torsion, we get the notion of malnormality.
Whereas the definitions of a relative automorphism and of a mapping-torus of a family of subgroups given in Definition 1.1 remain valid for injective endomorphisms, the definition of relative hyperbolicity for automorphisms is easily adapted to the more general case of injective endomorphisms:

**Definition 6.19.** Let $G$ be a finitely generated group and let $\mathcal{H}$ be a finite family of subgroups of $G$. An injective endomorphism $\alpha$ of $G$ is hyperbolic relative to $\mathcal{H}$ if and only if $\alpha$ is a relative endomorphism of $(G, \mathcal{H})$ and there exist $\lambda > 1$ and $M, N \geq 1$ such that, for any $w \in \text{Im}(\alpha^N)$ with $|w|_\mathcal{H} \geq M$, if $|\alpha^N(w)|_\mathcal{H} \geq \lambda|w|_\mathcal{H}$ does not hold then $w = \alpha^N(w')$ with $|w'|_\mathcal{H} \geq \lambda|w|_\mathcal{H}$.

**Theorem 6.20.** Let $G$ be a finitely generated group, let $\alpha$ be an injective endomorphism of $G$ and let $G_\alpha$ be the associated mapping-torus group, i.e. the associated ascending HNN-extension. Let $\mathcal{H}$ be a finite family of infinite subgroups of $G$ such that $\alpha$ is hyperbolic relative to $\mathcal{H}$. Assume that $\text{Im}(\alpha)$ is almost malnormal relative to $\mathcal{H}$. Then, if $G$ is strongly hyperbolic relative to $\mathcal{H}$, $G_\alpha$ is weakly hyperbolic relative to $\mathcal{H}$ and strongly hyperbolic relative to the mapping-torus of $\mathcal{H}$.

The reader will notice at once that the above theorem does not treat the extension of weakly relatively hyperbolic groups. The reason is that the condition of relative almost malnormality does not imply in this case the relative hallways-flare property. This last property is however also a necessary condition, although we do not give here a direct proof: in the absolute hyperbolicity case, Gersten was the first to give the converse to the combination theorem, using cohomological arguments [18] and we adapt his arguments in [15]. Bowditch exposed a more direct proof in [6].

**Proof of Theorem 6.20:** Before stating a first lemma let us recall that if $h$ is a geodesic in a Gromov hyperbolic space then $P_h(.)$ denotes a quasi projection on $h$.

**Lemma 6.21.** Let $G = \langle S \rangle$ be a finitely generated group which is strongly hyperbolic relative to a finite family of subgroups $\mathcal{H}$. There exists $C > 0$ such that if $K$ is a finitely generated subgroup of $G$ satisfying the following properties:

- it is almost malnormal relative to $\mathcal{H}$,
- it is strongly hyperbolic relative to a (possibly empty) finite family $\mathcal{H'}$ the subgroups of which are conjugated to subgroups in $\mathcal{H}$,
- $(K, |.|_{\mathcal{H}'})$ is quasi isometrically embedded in $(G, |.|_{\mathcal{H}})$,

and if $x, y$ (resp. $z, t$) are any two vertices in a same left-class $gK$ (resp. $hK$) with $g \neq h$ then $d_{\Gamma^S(G)}(P_{[z,t]}(x), P_{[z,t]}(y)) \leq C$.

**Proof:** In order to simplify the notations we write $d_{\mathcal{H}}(\cdot, \cdot)$ for $d_{\Gamma^S(G)}(\cdot, \cdot)$. Since $\Gamma^S(G)$ is hyperbolic, there is a constant $\delta \geq 0$ such that the geodesic triangles of are $\delta$-thin, and geodesic rectangles are $2\delta$-thin. This implies the existence of a quadruple of vertices $x_0, y_0, z_0, t_0$ with $x_0, y_0 \in [x, y]$, $z_0, t_0 \in [z, t]$ and $d_{\mathcal{H}}(x_0, z_0) \leq 2\delta + 1$, $d_{\mathcal{H}}(y_0, t_0) \leq 2\delta + 1$. Since $(K, |.|_{\mathcal{H}'})$ is $(\lambda, \mu)$-quasi isometrically embedded in $(G, |.|_{\mathcal{H}})$, and $\Gamma^S(G)$ is $\delta$-hyperbolic, there exist $c_0(\lambda, \mu, \delta)$ and $x_1, y_1, z_1, t_1$ such that $g^{-1}x_1, g^{-1}y_1 \in K$, $h^{-1}z_1, h^{-1}t_1 \in K$ and $d_{\mathcal{H}}(x_0, x_1) \leq c_0(\lambda, \mu, \delta)$, $d_{\mathcal{H}}(y_0, y_1) \leq c_0(\lambda, \mu, \delta)$, $d_{\mathcal{H}}(z_0, z_1) \leq c_0(\lambda, \mu, \delta)$, $d_{\mathcal{H}}(t_0, t_1) \leq c_0(\lambda, \mu, \delta)$. We choose $x_1, y_1, z_1, t_1$ to minimize the distance in $\Gamma^S(G)$ (that is the distance associated to the given finite set of generators $S$ of $G$) respectively to $x_0, y_0, z_0, t_0$. We denote by $[x_1, y_1]_K$ (resp. $[z_1, t_1]_K$) the images, under the embedding of $K$ in $G$, of geodesics between the pre-images of $x_1, y_1$ (resp. $z_1, t_1$) in $K$. Both $[x_1, y_1]_K$ and $[z_1, t_1]_K$ are $(\lambda, \mu)$-quasi geodesics. Moreover $[x_1, z_1][z_1, t_1][t_1, y_1]$ is a $(\lambda, 4\delta + 2 + 4c_0(\lambda, \mu, \delta) + \mu)$-quasi geodesic between $x_1$ and $y_1$. Since $G$ is strongly hyperbolic relative to $\mathcal{H}$, $\Gamma^S(G)$
satisfies the BCP property with respect to \( \mathcal{H} \). This gives a constant \( c_1(\lambda, \mu, \delta) \) such that the \( \mathcal{H} \)-classes \( [x_1, z_1] \) and \( [t_1, y_1] \) go through correspond to geodesics in \( \Gamma_S(G) \) with length smaller than \( c_1(\lambda, \mu, \delta) \): indeed, since \( x_1, y_1, z_1, t_1 \) were chosen to minimize the distances in \( \Gamma_S(G) \) with respect to \( x_0, y_0, z_0, t_0 \), the \( \mathcal{H} \)-classes crossed by \( [x_1, z_1] \) and \( [t_1, y_1] \) are not crossed by \( [x_1, y_1]_K \). Therefore the distance in \( (G, S) \) between \( x_1 \) and \( z_1 \) on the one hand, and between \( y_1 \) and \( t_1 \) on the other hand is less or equal to \((2\delta + 1) + 2c_0(\lambda, \mu, \delta))c_1(\lambda, \mu, \delta) \).

There are a finite number of elements in \( G \) with such an upper-bound on the length, measured with a word-metric associated to a finite set of generators. Whence, by the almost normality of \( K \) relative to \( \mathcal{H} \), an upper-bound on the length between \( x_1 \) and \( y_1 \), and so also between \( x_0 \) and \( y_0 \). Lemma 6.21 is proved.

From Lemma 6.21, the overlapping of two distinct left \( \text{Im}(\alpha) \)-classes is bounded above by a constant. Together with the fact that \( \alpha \) is a relatively hyperbolic endomorphism, this implies the exponential separation property. Getting the strong version of this property is done as in the proof of the strong relative hyperbolicity case of Theorem 1.5. Theorem 6.20 now follows from Theorem 6.5.

**Proof of Theorem 1.12:** It suffices to prove that the given conditions imply the strong hallways-flare property. Since we already proved in the proof of Theorem 1.9 that the first condition implies the hallways-flare property, it only remains to prove that any two exceptional orbits separate exponentially one from the other. But this is exactly what is required.

We conclude this list of statements by going back to 3-manifolds. Let us recall that a **closed 3-manifold** is a connected, compact 3-manifold without boundary. A **Seifert fibred space** is a connected, compact, orientable, irreducible 3-manifold which is a union of disjoint circles \( C_\alpha \), called the **fibers** of \( M^3 \), such that each \( C_\alpha \) admits a neighborhood \( T(C_\alpha) \), homeomorphic by a fiber-preserving homeomorphism \( h_\alpha \) to a **fibred solid torus** \( \mathbb{T}^2_{p,q} \), i.e. the suspension \( D^2 \times [0,1]/(x,1) \sim (r(x),0) \) of a rotation \( r \) of the disc \( D^2 \) centered at the origin \( O \) and of angle \( \frac{2\pi p}{q} \). The fibers of \( \mathbb{T}^2_{p,q} \) are the orbits of the rotation \( r \). The homeomorphism \( h_\alpha \) is required to carry \( C_\alpha \) to the \( r \)-orbit of \( O \). A **graph-manifold** is a compact, orientable, irreducible 3-manifold \( M^3 \) which admits a finite union of incompressible tori \( T_1, \cdots, T_r \) such that the closure of each connected component of \( M^3 \setminus \bigcup_{i=1}^r T_i \) is a Seifert-fibred manifold. By the JSJ-decomposition [22], given any closed, irreducible, orientable 3-manifold \( M^3 \) there exist a family of maximal graph-submanifolds \( \mathcal{G}M_1, \cdots, \mathcal{G}M_r \) in \( M^3 \) such that the closure of each connected component of \( M^3 \setminus \bigcup_{i=1}^r \mathcal{G}M_i \) is a compact 3-manifold with hyperbolic, finite volume interior, the boundary of which is a union of tori.

**Corollary 6.22.** Let \( M^3 \) be a closed, irreducible, orientable 3-manifold. Then the fundamental group of \( M^3 \) is strongly hyperbolic relative to a family formed by

(a) the subgroups \( G_i \) corresponding to certain conjugates of the fundamental groups of the maximal graph-submanifolds \( \mathcal{G}M_1, \cdots, \mathcal{G}M_r \) in \( M^3 \),

(b) the \( \mathbb{Z} \oplus \mathbb{Z} \)-subgroups corresponding to the incompressible tori in \( M^3 \setminus \bigcup_{i=1}^r \mathcal{G}M_i \).

**Proof.** The fundamental group of \( M^3 \) is the fundamental group of a graph of groups \( \mathcal{G} \) satisfying the following properties:

- the vertex groups are the \( G_i \)’s, together with the fundamental groups \( H_j \) of finite volume hyperbolic 3-manifolds with cusps,
- the edge groups are \( \mathbb{Z} \oplus \mathbb{Z} \)-subgroups,
• two vertex groups \( G_i \) and \( G_j \), \( i \neq j \), are not adjacent.

The graph \( G \) becomes a graph of strongly relatively hyperbolic groups when considering each edge-group and each vertex-group \( G_i \) strongly hyperbolic relative to itself, whereas each vertex-group \( H_j \) is considered as a group strongly hyperbolic relative to the \( \mathbb{Z} \oplus \mathbb{Z} \)-subgroups of the cusps [11] (i.e. boundary components). It is in fact a fine graph of strongly relatively hyperbolic groups because the cusp subgroups are malnormal. This last property gives the following stronger assertion: if \( T \) is the universal covering of \( G \) then for any \( G \)-tree of spaces \((X,T,\pi)\), for any \( v \geq 0 \) there is a uniform bound \( M \) on the vertical length of the \( v \)-corridors. In other words, if \( C \) is a \( v \)-corridor in \((X,T,\pi)\) then \( \pi(C) \) has diameter smaller than \( M \) in \( T \). It follows from Theorem 6.5 that the fundamental group of \( G \) is strongly hyperbolic relative to a family of subgroups as given by Corollary 6.22.

An alternative proof is obtained by combining [11] (any hyperbolic 3-manifold with boundary tori is strongly hyperbolic relative to the boundary subgroups) and the combination theorem of [10].

7. Proof of Proposition 3.9

Conventions: The constants of hyperbolicity and of quasi isometry are chosen sufficiently large enough to satisfy the conclusions of Lemma 2.19, and also sufficiently large enough so that computations make sense. Moreover the horizontal subsets of the \((a,b)\)-quasi geodesics considered will be assumed to be horizontal geodesics. The hyperbolicity of the strata gives, for any \( a \geq 1 \) and \( b \geq 0 \), a positive constant \( C(a,b) \) such that any \((a,b)\)-quasi geodesic \( g \) may be substituted by another one \( g' \) with \( d_X^H(g,g') \leq C(a,b) \) and satisfying this latter property.

In the proofs of the various intermediate statements, when referring to a constant provided by an earlier result we will sometimes indicate between parentheses the values of some of the parameters from which it depends.

Our first lemma is about quasi geodesics. It holds not only in a corridor but in the whole tree of hyperbolic spaces.

**Lemma 7.1.** Let \((\tilde{X},T,\pi)\) be a tree of hyperbolic spaces which satisfies the hallways-flare property. For any \( a \geq 1 \), \( b \geq 0 \) and for any \( v \geq D_{A_A} \) there exist \( C \geq 0 \) and \( D \geq 0 \) such that, if \( g \) is a \((a,b)\)-quasi geodesic in \( \tilde{X} \), if \([x,y] \subset g \cap X_a\) satisfies \( d_{\text{hor}}(x,y) \geq C \) then for any \( T \)-geodesic \( \omega \) starting at \( \alpha \) with \( |\omega|_T \geq D + nt_0 \), \( n \geq 1 \), we have \( d^\alpha_{\text{hor}}(\omega x,\omega y) \geq \lambda^n d_{\text{hor}}(x,y) \).

**Proof:** We denote by \( \lambda > 1 \), \( M, t_0 \geq 1 \) the constants of hyperbolicity and by \( \lambda_+, \mu \) the constants of quasi isometry. Let us choose \( n_\ast(a) \) such that \( \frac{a}{\lambda^2} < 1 \). Solving the inequality \( e > a(\frac{1}{\lambda^2} e + 2n_\ast t_0) + b \) gives us \( e(a,b) \geq \frac{2a_0t_0+b}{1-a_0\frac{1}{\lambda^2}} \).

**Claim:** If \( d_{\text{hor}}(x,y) \geq e(a,b) \), if \( x', y' \) are the endpoints of two \( v \)-vertical segments \( s, s' \) of vertical length \( nt_0 \), starting at \( x \) and \( y \) and with \( \pi(s) = \pi(s') \), then for any \( T \)-geodesic \( \omega_0 \) such that \( \omega_0 \pi(s) \) is a \( T \)-geodesic and \( |\omega_0|_T = t_0 \), \( d^\lambda_{\text{hor}}(\omega_0' x', \omega_0' y') \geq \lambda d_{\text{hor}}(x', y') \) holds.

**Proof of Claim:** Assume the existence of \( \omega \) with \( |\omega|_T = n_\ast t_0 \) such that for some \( x', y' \) with \( x \in \omega x', y \in \omega y' \) and \( d_{\text{hor}}(x', y') \geq M, d_{\text{hor}}(x,y) \geq \lambda t_0 d_{\text{hor}}(x', y') \) holds. Then \( \frac{1}{\lambda^2} e + 2n_\ast t_0 \) is the telescopic length of a telescopic chain between \( x \) and \( y \). But the inequality given at the beginning of the proof tells us that the existence of such a telescopic chain is a contradiction with the fact that \( g \) is a \( v \)-telescopic \((a,b)\)-quasi geodesic. Therefore, if \( d_{\text{hor}}(x,y) \geq e(a,b) \) and \( d_{\text{hor}}(x,y) \geq \lambda_+^n (M + \mu) \) (this last inequality is to assert that...
\(d_{\text{hor}}(x', y') \geq M\) - see above), then \(d_{\text{hor}}(x', y')\) does not increase after \(t_0\) in the direction of the \(v\)-vertical segments \(s, s'\). The claim follows from the exponential separation of the \(v\)-vertical segments.

From the inequality given by the Claim, since \(d_{\text{hor}}(x', y') \geq \lambda^n + \mu\), we easily compute an integer \(N_\ast\) such that, if \(\omega_0\) is as in the Claim but with length \(N_\ast t_0\) then \(d_{\text{hor}}^n(\omega_0 \pi(s)) x, [\omega_0 \pi(s)] y \geq \lambda d_{\text{hor}}(x, y)\). Setting \(D = N_\ast t_0\) and \(C(a, b) = c(a, b)\), the constant computed above, we get the lemma. \(\square\)

**Notations:** \(\delta\) a fixed non negative constant, \((\hat{X}, \mathcal{T}, \pi)\) a tree of \(\delta\)-hyperbolic spaces, \(w \geq D_{4.4}\) and \(v \geq C_{3.6}(w)\) two constants, \(\lambda > 1, M, t_0 \geq 1\) the associated constants of hyperbolicity, \(\lambda_+, \mu\) the associated constants of quasi isometry.

**Lemma 7.2.** For any \(a \geq 1, b \geq 0\), there exists \(C \geq 0\) such that if \(C\) is a generalized \(w\)-corridor with exponentially separated \(v\)-vertical segments, if \(g\) is a \(v\)-telescopic chain which is a \((a, b)\)-quasi geodesic of \((C, d_{\text{tel}})\), if the endpoints \(x, y\) of \(g\) both lie in a same stratum \(X_\alpha\), if \(d_{\text{hor}}(x, y) \geq C\) then, for any \(T\)-geodesic \(\omega\) starting at \(\alpha\) with \(|\omega|_T \geq C + nt_0\), \(n \geq 1\), and \(\omega \cap \pi(g) = \{\alpha\}\), we have:

\[d_{\text{hor}}(\omega x, \omega y) \geq \lambda^n d_{\text{hor}}(x, y)\]

**Proof:** Let us observe that, if \([p, q]\) is any horizontal geodesic in \(g\) then the \(v\)-vertical trees of \(p\) and \(q\) bound a horizontal geodesic \([p', q']\) in \([x, y]\).

**Claim:** If \(d_{\text{hor}}(p', q') \geq C t_0\) with \(C t_0 := \lambda^n (C_{7.1} + t_0 + \mu)\) then for any \(\omega\) as given by the current Lemma with \(|\omega|_T \geq D_{7.1} + t_0\), \(d_{\text{hor}}^n(\omega p', \omega q') \geq \lambda d_{\text{hor}}(p', q')\).

**Proof of Claim:** If \(p'\) and \(q'\) are not exponentially separated in the direction of \(p, q\) after \(t_0\), then, because of the hallways-flare property, they are exponentially separated after \(t_0\) in the direction of \(\omega\), which yields the announced inequality. Let us assume that \(p', q'\) are separated after \(t_0\) in the direction of \([\pi(p'), \pi(p)]\). Thus \(d_{\text{hor}}^n(rp', rq') \geq \lambda^n d_{\text{hor}}(p', q')\) for a \(T\)-geodesic \(r\) with \(|r|_T = nt_0\) and \(r \cap \omega = \{\alpha\}\). Therefore \(d_{\text{hor}}(p, q) \geq C_{7.1} + t_0\). Lemma 7.1 then implies that \(p, q\) are exponentially separated in the direction of \([\pi(p), \pi(p')]\) after \(D_{7.1} + t_0\), and the claim is proved.

There is a finite decomposition of \([x, y] \subset X_\alpha\) in subgeodesics \([p_j', q_j']\) with disjoint interiors such that each \([p_j', q_j']\) connects two \(v\)-vertical trees through the endpoints of a maximal horizontal geodesic in \(g\). We denote by \(I_D\) the set of \([p_j', q_j']\)'s with \(d_{\text{hor}}(p_j', q_j') \geq C t_0\) and by \(I_C\) the set of the others. Let us choose an integer \(n \geq 1\). We consider a stratum \(X_\beta\) with \(d_T(\beta, \alpha) = D_{7.1} + nt_0\). Let \(h\) be the horizontal geodesic in \(C \cap X_\beta\) which connects the two \(v\)-vertical trees through \(x\) and \(y\). Assume that the endpoints of \(h\) are exponentially separated after \(t_0\) in the direction of \([\beta, \alpha]\). Then:

\[(1) \quad \lambda^n |I_D|_{\text{hor}} \leq |h|_{\text{hor}} \leq \lambda^{-n}(|I_D|_{\text{hor}} + |I_C|_{\text{hor}})\]

so that

\[|I_C|_{\text{hor}} \geq \frac{\lambda^n - \lambda^{-n}}{\lambda^{-n}} |I_D|_{\text{hor}}\]

and consequently, since \(d_{\text{hor}}(x, y) = |I_D|_{\text{hor}} + |I_C|_{\text{hor}}\),

\[|I_C|_{\text{hor}} \geq \frac{X(n)}{1 + X(n)} d_{\text{hor}}(x, y)\]
with $X(n) = \frac{\lambda^n - \lambda^{-n}}{\lambda^n}$. Since $\lim_{n \to +\infty} \frac{X(n)}{1 + X(n)} = 1$, there is $n_* \geq 0$ such that for any $n \geq n_*$,

$$|I_C|_{\text{hor}} \geq \frac{1}{2} d_{\text{hor}}(x, y).$$

But, by definition, the horizontal length of each subgeodesic in $I_C$ is smaller than $Cte$. Thus the number of elements in $I_C$ is at least the integer part of $\frac{1}{2Cte} d_{\text{hor}}(x, y) + 1$. Furthermore, since $g$ is a $v$-telescopic chain, the telescopic length of any subset of $g$ containing $j$ maximal horizontal geodesics is at least $(j - 1)$. We so obtain:

$$|g|_{\text{tel}}^v \geq \frac{1}{2Cte} d_{\text{hor}}(x, y).$$

On the other hand:

$$d_{\text{tel}}^v(x, y) \leq \lambda^{-n} d_{\text{hor}}(x, y) + 2nt_0,$$

since there is a $v$-telescopic chain between $x$ and $y$ the telescopic length of which is given by the right-hand side of the above inequality. Since $g$ is a $(a, b)$-quasi geodesic, the last two inequalities give $n_* \geq 0$ such that for $n \geq n_*:

$$d_{\text{hor}}(x, y) \leq \frac{2ant_0 + b}{2Cte - a\lambda^{-n}}.$$

Taking the maximum of $n_*, n_*$, and the above upper-bound for $d_{\text{hor}}(x, y)$, we get the announced constant in the case where the endpoints of the horizontal geodesic $h$ above are exponentially separated in the direction of $[\beta, \alpha]$. If not, there are in all the other directions so that we easily get a constant $N \geq 0$ such that $d_{\text{hor}}^h(\omega x, \omega y) \geq \lambda d_{\text{hor}}(x, y)$ for any $T$-geodesic $\omega$ with $|\omega|_T = Nt_0$ and $[\pi(x), \pi(h)] \subset \omega$. Lemma 7.2 is then easily deduced. □

As a consequence we have:

**Corollary 7.3.** For any $a \geq 1, b \geq 0$ and $d \geq M$, there exists $C \geq d$ such that if $C$ is a generalized $v$-corridor with exponentially separated $v$-vertical segments, if $g$ is any $v$-telescopic chain which is a $(a, b)$-quasi geodesic of $(C, d^v_{\text{tel}})$, if $x, y$ are the endpoints of two $v$-vertical segments $s, s'$ over a same edge-path in $T$, with $\pi(s) \cap \pi(g) = \{\alpha\}$ and such that $d_{\text{hor}}^h(s, s') \leq d$, then $d_{\text{hor}}(x, y) \leq C$.

**Remark 7.4.** At this point, we would like to notice that Lemma 7.2 is similar to Lemma 6.7 of [13]. However in addition of some misprints, a slight mistake took place there in the proof of the Lemma. Indeed the inequality (1) in the proof of Lemma 7.2 is true here, in the generalized corridor, but there the constant $\lambda$ should have been modified to take into account the so-called “cancellations”.

**Lemma 7.5.** For any $r \geq 0$, there exists $C \geq 0$ such that if $C$ is a generalized $v$-corridor with exponentially separated $v$-vertical segments, if $x$ and $y$ are the endpoints of a $r$-vertical segment $s$ in $C$, if the intersection-point $z$ of some $v$-vertical tree through $y$ in $C$ with the stratum $X_{\pi(x)}$ satisfies $d_{\text{hor}}(x, z) \geq C$, then for any $T$-geodesic $\omega$ with $|\omega|_T = nt_0$, $n \geq 1$, and $\omega \cap \pi(s) = \{\pi(x)\}$, $d_{\text{hor}}^h(\omega x, \omega z) \geq \lambda^n d_{\text{hor}}(x, z)$.

**Proof:** If $|s|_{\text{vert}} \leq t_0$, the existence of the constants of quasi isometry, Item (a) of Lemma 2.11, and the definition of a $r$-vertical segment give an upper-bound for $d_{\text{hor}}(x, z)$. Let us thus assume $|s|_{\text{vert}} > t_0$. Choose $d$ such that $\lambda d - r' \geq 2r'$, where $r'$ is the above upper-bound when $|s|_{\text{vert}} = t_0$. Then set $C = \max(d, M)$. Assume that $d_{\text{hor}}(x, z) \geq C$ and that $x$ and $z$ are exponentially separated in the direction given by $s$. If $[\pi(x), \pi(y)] = \omega_0\omega'$ with $|\omega_0|_T = t_0$, then $d_{\text{hor}}^h(\omega_0 x, \omega_0 z) \geq \lambda d_{\text{hor}}(x, z)$. Thanks to the inequality used to
define $d$, one easily concludes that the horizontal distance between $s$ and the vertical tree through $y$ increases along $s$ when going from $x$ to $y$ which of course cannot happen. The conclusion follows from the hallways-flare property. □

**Proof of Proposition 3.9:** We are given a $w$-corridor $C$, $L$ the horizontal distance between two points $x$ and $y$ in $C$, and $g$ a $(a,b)$-quasi geodesic in $(C,d^a_{tel})$ from a $v$-vertical tree through $x$ to a $v$-vertical tree through $y$ with $v \geq C_{3.6}(w)$. We assume that the $v$-vertical segments in $C$ are exponentially separated. We consider the region $R$ with vertical width $C_{7.3}$ centered at the stratum $X_\alpha$ with $\alpha = \pi(x)$. We decompose $g$ in three subsets: the first one, denoted $g_0$, from the initial point of $g$ until the first point $z$ in $g \cap R$, the second one, denoted $g_1$, from $z$ to the last point $t$ in $g \cap R$, the third one, denoted $g_2$, from $t$ to the terminal point of $g$. Obviously $g_1$ can be approximated by the concatenation of two vertical segments with a horizontal geodesic in $X_\alpha$ (the approximation constant only depend on $L, a$ and $b$). We denote by $g_1'$ the resulting set.

We now consider a maximal chain in $g_0$ which satisfies the following properties:

- its endpoints lie in a same stratum $X_\beta$,
- its image under $\pi$ does not intersect $[\alpha, \beta]$.

From Corollary 7.3, the endpoints of such a subchain are at horizontal distance smaller than $C_{7.3}$ one to each other. Thus, by substituting each such subchain by a horizontal geodesic connecting its endpoints, we construct a $C_{7.3}$-vertical segment $g_0'$. We do the same thing for $g_2$, so obtaining a $C_{7.3}$-vertical segment $g_2'$. From Lemma 7.5, $g' = g_0' \cup g_1' \cup g_2'$ lies in a bounded neighborhood of the $v$-vertical segments connecting its endpoints to $x_1$ and $x_2$. From the construction, $d^v_{tel}(g, g') \leq aC_{7.3} + b + 1$. The proposition follows. □

### 8. Quasiconvexity of corridors

In this section we prove Proposition 3.10, its adaptation to generalized corridors and Proposition 6.14.

#### 8.1. Two basic lemmas

We need first a very general lemma about Gromov hyperbolic spaces.

**Lemma 8.1.** Let $(X, d)$ be a Gromov hyperbolic space. There exists $C \geq 0$ such that for any $r \geq C$ there is $D \geq 0$, increasing and affine in $r$, such that if $[x,y]$ is a diameter of a ball $B_{x_0}(r)$, if $\omega$ is any chain in $X$ with $\omega \cap B_{x_0}(r) = \{x,y\}$, then $|\omega|_d \geq e^D$.

This lemma is a rewriting of Lemma 1.6 of [9]. □

**Lemma 8.2.** Let $X$ be a tree of $\delta$-hyperbolic spaces which satisfies the hallways-flare property. For any $v \geq D_{1.4}$, there exists $C \geq 0$ such that if $x, y, z, t$ are the vertices of a geodesic quadrilateral in some stratum $X_\alpha$, with $d_{hor}(x,z) \leq 2\delta$, $d_{hor}(y,t) \leq 2\delta$, and $d_{hor}(x,y) \geq C$, $d_{hor}(z,t) \geq C$, then for any $T$-geodesic $\omega$ with $|\omega|_T \geq C_{2.19} + n_t0$ and starting at $\pi(x)$, when considering the $v$-vertical segments over $\omega$ we have:

$$d^v_{hor}(\omega x, \omega y) \geq \lambda^n d_{hor}(x,y) \Rightarrow d^v_{hor}(\omega z, \omega t) \geq \lambda^n d_{hor}(z,t)$$

**Proof:** If $A, B$ are two subsets of a metric space $(X, d)$, we set $d^v(A, B) = \sup_{x \in A, y \in B} d(x, y)$. Let us consider any $T$-geodesic $\omega$ with $|\omega|_T = t_0$ starting at $\alpha$. From Lemma 2.11,

$$d^v_{hor}(\omega x, \omega z) \leq \lambda^v_{t_0}(2\delta + \mu)$$
and
\[ d_{\text{hor}}(\omega y, \omega t) \leq \lambda t_0 (2\delta + \mu). \]
Assume \( d_{\text{hor}}(\omega x, \omega y) \geq \lambda d_{\text{hor}}(x, y) \) but \( d_{\text{hor}}(\omega z, \omega t) < \lambda d_{\text{hor}}(z, t) \).

We take \( d_{\text{hor}}(x, y) \geq M \) and \( d_{\text{hor}}(z, t) \geq M \). Assume \( d_{\text{hor}}(\omega z, \omega t) \leq \frac{1}{\lambda} d_{\text{hor}}(z, t) \). But
\[ d_{\text{hor}}(z, t) \leq 4\delta + d_{\text{hor}}(x, y). \]
Putting together these inequalities we get
\[ \lambda d_{\text{hor}}(x, y) \leq 2\lambda t_0 (2\delta + \mu) + \frac{1}{\lambda} (4\delta + d_{\text{hor}}(x, y)). \]

Whence an upper bound for \( d_{\text{hor}}(x, y) \) and thus for \( d_{\text{hor}}(z, t) \). If \( d_{\text{hor}}(\omega z, \omega t) > \frac{1}{\lambda} d_{\text{hor}}(z, t) \) then the lemma follows from the definition of the constant \( C_{2,19} \), see the corresponding lemma. \( \square \)

The above two lemmas are not needed if one only considers trees of 0-hyperbolic spaces, the proof in this last case being much simpler.

8.2. Approximation of quasi geodesics with bounded vertical deviation.

Lemma 8.3 below states that in a tree of hyperbolic spaces \((\tilde{X}, T)\) a quasi geodesic with bounded image in \( T \) lies close to a corridor between its endpoints. This is intuitively obvious and nothing is new neither surprising in the arguments of the proof: they heavily rely upon the \( \delta \)-hyperbolicity of the strata and the fact that strata are quasi isometrically embedded into each other. For the sake of brevity, we do not develop them here.

**Lemma 8.3.** Let \((\tilde{X}, T, \pi)\) be a tree of hyperbolic spaces. For any \( \kappa, b \geq 0, a \geq 1 \) and \( v \geq D_{4,4} \) there exists \( C \geq 0 \) such that if \( g \) is any \((a, b)\)-quasi geodesic of \( \tilde{X} \) with \( \text{diam}_T(\pi(g)) \leq \kappa \), if \( C \) is a generalized \( v \)-corridor whose vertical boundaries pass through the endpoints of \( g \) then \( g \subset N^C_{\tilde{X}}(C) \).

8.3. Stairs. Notations: The sign \( \simeq_1 \) stands for an equality up to \( \pm 1 \), \((\tilde{X}, T, \pi)\) a tree of hyperbolic spaces which satisfies the hallways-flare property, \( v \geq D_{4,4} \) a constant.

**Definition 8.4.** Let \( r \geq M \). A \( r \)-stair relative to a generalized \( v \)-corridor \( C \) is a \( v \)-telescopic chain \( S \) the maximal vertical segments of which have vertical length greater than \( C_{2,19} \) and such that, for any maximal horizontal geodesic \([a_i, b_i]\) in \( S \):

(a) \( d_{\text{hor}}(a_i, b_i) \geq r \) and \( d_{\text{hor}}([a_i, b_i], C) \simeq_1 d_{\text{hor}}(a_i, P^C_{\text{hor}}(a_i)) \),

(b) any two points \( a, b \in [a_i, b_i] \) with \( d_{\text{hor}}(a, b) \geq r \) are exponentially separated in the direction of the \( T \)-geodesic \([\pi(a_i), \pi(a_{i+1})]\).

See Figure 10.
Lemma 8.5. With the notations of Definition 8.4: there exist $C \geq C_{8.2}$ such that for any $r \geq C$, if $C$ is a generalized $v$-corridor, if $S$ is a $r$-stair relative to $C$, if $U$ is a generalized $v$-corridor between a vertical tree through the terminal point of $S$ and a vertical boundary of $C$, then

$$S \subset N_{\text{hor}}^{r+2\delta}(U).$$

Proof: Let $a_i, b_i \in S$ as given in Definition 8.4 and let $z$ be a point at the intersection of the stratum $X_{(a_i)}$ with a vertical tree through some point farther in the stair. Then:

Claim 1: There exists $K > 0$ not depending on $a_i$ nor $z$ such that, if $r$ is sufficiently large enough then $d_{\text{hor}}([a_i, z], C) \geq d_{\text{hor}}(a_i, P_{C}^{\text{hor}}(a_i)) - K$.

Proof of Claim 1: Choose $K$ such that $e^{D_{8.1}(K)} > 4\delta + 1$ and assume $d_{\text{hor}}([a_i, z], C) < d_{\text{hor}}(a_i, P_{C}^{\text{hor}}(a_i)) - K$. Then Lemma 8.1 implies that $[b_i, z]$ descends at least until a $2\delta$-neighborhood of $a_i$. Assume $r \geq C_{8.2} + 2\delta$. Then Lemma 8.2 gives an initial segment of $[b_i, z]$ of horizontal length greater than $r - 2\delta$ which is dilated in the direction of $[\pi(a_i), \pi(a_{i+1})]$. If $r$ is chosen sufficiently large enough with respect to the constants of hyperbolicity for a corridor (see Lemma 2.18), we get $z'$ at the intersection of the considered vertical tree through $z$ with the stratum $X_{\pi(a_{i+1})}$ such that $d_{\text{hor}}([a_{i+1}, z'], C) < d_{\text{hor}}(a_{i+1}, P_{C}^{\text{hor}}(a_{i+1})) - K$. The repetition of these arguments show that the horizontal distance between $S$ and the vertical tree through $z$ does not decrease along $S$. This is an absurdity since $z$ was chosen in a vertical tree through a point farther in $S$. The proof of Claim 1 is complete.

Claim 2: There exists $K(r)$ not depending on $b_i$ nor $z$ such that, if $r$ is sufficiently large enough then $d_{\text{hor}}([b_i, z], C) \geq d_{\text{hor}}(b_i, P_{C}^{\text{hor}}(b_i)) - K(r)$.

Proof of Claim 2: Let $z_* \in [b_i, z]$ with $d_{\text{hor}}(z_*, P_{C}^{\text{hor}}(z_*)) \simeq_{1} \max(d_{\text{hor}}([b_i, z], C), d_{\text{hor}}(a_i, P_{C}^{\text{hor}}(a_i)))$. From the $\delta$-hyperbolicity of the strata, $[b_i, z_*]$ lies in the horizontal $2\delta$-neighborhood of $[a_i, b_i]$. Assume $d_{\text{hor}}(b_i, z_*) \geq r$ and is sufficiently large enough to apply Lemma 8.2. Then there is $K(r)$ such that, if $z_*$ satisfies $d_{\text{hor}}(z_*, P_{C}^{\text{hor}}(z_*)) < d_{\text{hor}}(b_i, P_{C}^{\text{hor}}(b_i)) - K(r)$, the points $b_i$ and $z_*$ are exponentially separated in the direction of $[\pi(a_i), \pi(a_{i+1})]$. We thus obtain at $a_{i+1}$ a situation similar to that of Claim 1. The proof of Claim 2 follows.

Lemma 8.5 is easily deduced from the above two claims, we leave the reader work out the easy details.

Lemma 8.6. For any $r \geq C_{8.5}$ there exists $C > 0$ such that, if $C$ is a generalized $v$-corridor, if $S$ is a $r$-stair relative to $C$ which is not contained in the vertical $C$-neighborhood of the stratum containing its initial point, then the terminal point of $S$ does not belong to the $r$-neighborhood of $C$ in $\bar{X}$.

Proof: Decompose $S$ in maximal substairs $S_0 \cdots S_k$ such that $\pi(S_j)$ is a geodesic of $T$. Let $[a_i, b_i]$ be the first maximal horizontal geodesic in $S_j$, let $x$ be the initial point of $S_j$ and let $z$ be any point in $S_j$ with $nt_0 \leq d_T(\pi(z), \pi(x)) \leq (n+1)t_0$.

The inequality

$$d_{\text{hor}}(z, P_{C}^{\text{hor}}(z)) \geq Cte\lambda^nd_{\text{hor}}(a_i, b_i)$$

is an easy consequence of the definition of a stair and of Lemma 8.2 as soon as $r \geq C_{8.2}$. Indeed, the initial segment of horizontal length $r$ in $[b_i, P_{C}^{\text{hor}}(b_i)]$ lies in the horizontal $2\delta$-neighborhood of $[b_i, a_i]$. The assertion then follows from Item (b) of Definition 8.4 and Lemma 8.2.

The inequality (2) readily gives the announced result.
8.4. Approximation of a quasi geodesic by a stair.

Notations: $(\tilde{T}, \mathcal{T})$ a tree of $\delta$-hyperbolic spaces which satisfies the hallways-flare property, $v \geq D_{4.4}$.

Lemma 8.7. For any $a \geq 1, b \geq 0$ there exists $D \geq 0$ such that for any $r \geq D$ there are $C, E \geq 0$, where $E$ is affine in $r$, such that if $\mathcal{C}$ is a generalized $v$-corridor, if the endpoints of a $v$-telescopic $(a, b)$-quasi geodesic $g$ are in a horizontal $r$-neighborhood of $\mathcal{C}$, if $g$ lies in the closed complement of this horizontal neighborhood and if the maximal vertical segments in $g$ have vertical length greater than $3(C_{2.19} + D_{7.1})$ then either $g$ lies in the $C$-neighborhood of a $E$-stair relative to $\mathcal{C}$ or $g$ is contained in the $C$-neighborhood of $\mathcal{C}$.

Proof: We decompose the proof in two steps. The first one is only a warm-up, to present the ideas in a particular, but important, case. The general case, detailed in the second step, is technically more involved but no new phenomenon appears.

Step 1: Proof of Lemma 8.7 when the horizontal length of any maximal horizontal path in $g$ is greater than some constant (depending on $a$ et $b$). The endpoints of any horizontal path $h$ in $g$ with horizontal length greater than $C_{7.1}$ are exponentially separated under every geodesic $\omega$ of $\mathcal{T}$ with length $D_{7.1}$. If $|h|_{\text{hor}} \geq C_{8.2}$, this is also true for any horizontal geodesic $h'$ in the $2\delta$-neighborhood of $h$. Finally, if $|h|_{\text{hor}}$ is sufficiently large enough, by Lemma 2.18 the endpoints of $h$ are also exponentially separated in any $v$-corridor containing $h$. If $(a, b)$ (we do not indicate the dependance on $v$) is the maximum of the above constants, we now assume $|h|_{\text{hor}} \geq 3e(a, b)$.

Let us consider two consecutive maximal horizontal geodesics $h_1, h_2$ in $g$, separated by a vertical segment $s$. Let $\mathcal{D}$ be a corridor containing $h_1$ and $s$. Then:

$$|h_2 \cap \mathcal{N}_{\text{hor}}(\mathcal{D})|_{\text{hor}} \leq e(a, b).$$

Otherwise we have a contradiction with the fact that the endpoints of any subgeodesic of $h_2$ whose length is greater than $C_{7.1}$ are exponentially separated in the direction of $h_1$.

From the inequality (3), the concatenation of $h_1$, $s$ and $h_2$ is $e(a, b)$-close, with respect to the horizontal distance, of a $2e(a, b)$-stair relative to $\mathcal{C}$ if $d_{\text{hor}}(h_1, \mathcal{C}) \simeq_1 d_{\text{hor}}(a_1, P_{\mathcal{C}}^{\text{hor}}(a_1))$ where $a_1$ is the initial point of $h_1$.

Let us now set $r \geq 3e(a, b)$ and assume that the maximal horizontal geodesics in $g$ have horizontal length greater than $r$. Let $x$ be the initial point of $g$ (in particular $d_{\text{hor}}(x, P_{\mathcal{C}}^{\text{hor}}(x)) \simeq_1 r$). Let $s$ be the vertical segment starting at $x$ and ending at $y$ in $g$. Let $h$ be the maximal horizontal geodesic following $s$ along $g$. Let $n \geq 1$ be the greatest integer with $n(C_{2.19} + D_{7.1}) \leq |s|_{\text{vert}}$.

By assumption $x$ and $P_{\mathcal{C}}^{\text{hor}}(x)$ are exponentially separated in the direction of $s$. Since the strata are quasi isometrically embedded one into each other, this gives $\kappa > 1$ such that, any two points $p, q \in [x, P_{\mathcal{C}}^{\text{hor}}(x)]$ with $d_{\text{hor}}(p, q) \geq \max(\frac{r}{\kappa}, M)$ satisfy $d_{\text{hor}}(\pi(s)p, \pi(s)q) \geq \lambda^n d_{\text{hor}}(p, q)$. Thus the same arguments as those exposed above when working with $h_1, h_2$ show that $|h \cap \mathcal{N}_{\text{hor}}[\pi(y, P_{\mathcal{C}}^{\text{hor}}(y))]|_{\text{hor}} \leq \max(e(a, b), \frac{1}{\lambda^n}, r, M)$. If $n$ is greater than some critical constant $n_*$, this last maximum is equal to $e(a, b)$. Thus, in this case we take $h_1 = [x, P_{\mathcal{C}}^{\text{hor}}(x)]$ and $h_2 = h$: the above arguments prove that the concatenation of $h_1, s$ and $h_2$ is $e(a, b)$-close to a $e(a, b)$-stair. If $n$ is smaller than $n_*$, then we substitute $r$ by $\lambda_{n_*}(C_{2.19} + D_{7.1})r$, modify $g$ by taking the starting point at the endpoint $y$ of $s$ and take $h_1$ as the first maximal horizontal geodesic.
In both cases, by repeating the arguments above at any two consecutive maximal horizontal geodesic following the first two ones along \( g \), we show that \( g \) is \( e(a, b) \)-close, with respect to the horizontal distance, of a \( e(a, b) \)-stair relative to \( C \).

\( \square \)

**Step 2: Adaptation of the argument to the general case:** The boundary trees of \( C \) are denoted by \( L_1 \) and \( L_2 \), and \( g \) goes from \( L_1 \) to \( L_2 \). We choose a positive constant \( r \), which when necessary will be set sufficiently large enough with respect to the constants \( C_{8,5} \), \( M, \delta \) and \( C_{8,2} \). Let \( x_0 \) be the initial point of \( g \). It lies in the boundary of the horizontal \( r \)-neighborhood of \( C \). We denote by \( C_i \) and \( x_i, i = 1, \ldots \), a sequence of corridors and points of \( g \) defined inductively as follows:

(a) \( C_i \) is a corridor with boundary trees a \( v \)-vertical tree through \( x_{i-1} \) and the \( v \)-vertical boundary \( L_2 \) of \( C \).

(b) \( x_i \) is the first point following \( x_{i-1} \) along \( g \) such that \( d_{\text{hor}}(x_i, P_{C_i}^{\text{hor}}(x_i)) \geq r \).

The chain in \( g \) between \( x_{i-1} \) and \( x_i \) is denoted by \( g_{i-1,i} \). Obviously \( g_{i-1,i} \) is contained in the horizontal \( r \)-neighborhood of \( C_i \). We project it to \( C_i \). From Lemma 3.12, we get a \( D_{3,12} \)-telescopic \((C_{3,12}, C_{3,12})\)-quasi geodesic of \((C_i, d_{\text{tel}}^{D_{3,12}})\). We set \( X(a, b, r) = C_{3,9}(r, C_{3,12}, C_{3,12}) \). From Proposition 3.9, \( P_{C_i}^{\text{hor}}(g_{i-1,i}) \) is contained in the \( X(a, b, r) \)-neighborhood of the concatenation of a subpath of \([x_{i-1}, P_{C_{i-1}}^{\text{hor}}(x_{i-1})]\) with a vertical segment in \( C_i \) (and is followed by \([P_{C_i}^{\text{hor}}(x_i), x_i]\)). Consider in this approximation of \( (a \text{ subchain of } g) \) a maximal collection of points \( y_i \) which defines a \( r \)-stair relative to \( C \). The points \( y_i \) do not necessarily agree with the \( x_i \)'s, because it might happen that, after \( x_{i-1} \) for instance, the approximation constructed above reenters in the \( r \)-neighborhood of \( C_{i-1} \) before leaving the \( r \)-neighborhood of \( C_i \). We proceed as in Step 1 and choose the \( y_i \)'s so that:

(a) either \( y_i \) is contained in a maximal horizontal geodesic, and from the observations in Step 1, this horizontal geodesic may be included in a stair,

(b) or the vertical distance from \( y_i \) to the next horizontal geodesic is at least \( C_{2,19} + D_{7,1} \).

Either we obtain a non-trivial \( r \)-stair relative to \( C \) which approximates a subchain \( g_0' \) of \( g \) or the approximation we constructed above exhausts \( g \) and is contained in some telescopic neighborhood of \( C \) the size of which is obtained from the previously exhibited constants. In this last case, the same assertion holds for the whole \( g \). This is one of the announced alternatives.

We can thus assume that we get \( y_0, \ldots, y_k \) forming a \( r \)-stair relative to \( C \). It is denoted by \( S \). Since the strata are quasi isometrically embedded one into each other, there is \( \kappa > 1 \), only depending on the constants of quasi isometry, such that \( S \) is in fact a \( \max(\frac{1}{\kappa}r, M, e(a, b)) \)-stair relative to \( C \). As soon as \( r > \kappa(M + e(a, b)) \), which we suppose from now, this maximum is just \( \frac{r}{\kappa} \). Thus \( S \) is a \( \frac{r}{\kappa} \)-stair whose maximal horizontal geodesics have horizontal length at least \( r \).

By construction \( S \) approximates \( g_0' \subset g \). We now consider the maximal subchain \( g'_1 \) of \( g \) starting at (or near - recall that we constructed an approximation of a subchain of \( g \) \( y_k \) which lies in the \( r \)-neighborhood of \( C_k \). This last corridor plays the rôle of the corridor \( U \) of Lemma 8.5. We project the subchain \( g'_1 \) to \( C_k \), so getting a \((C_{3,12}, C_{3,12})\)-quasi geodesic of this corridor. From Lemma 8.5, and because of the hyperbolicity of the strata, each horizontal geodesic of the \( \frac{r}{\kappa} \)-stair \( S \) admits a subgeodesic with horizontal length greater than \( \frac{r}{\kappa} \) in the horizontal \( 2\delta \)-neighborhood of \( C_k \). If \( r \) is chosen sufficiently large enough, Lemma 8.2 gives horizontal geodesics in \( C_k \) with horizontal length greater than \( M \) which are dilated in the same directions than the horizontal geodesics of \( S \). Now Proposition
3.9 applies and allows us to approximate the projection of \( g'_1 \) on \( C_k \) by a sequence of these horizontal geodesics. But each one of these horizontal geodesics is close to a point in \( g'_0 \subset g \). Thus, since \( g \) is a \((a, b)\)-quasi geodesic, the vertical length of \( g'_1 \), and so its telescopic length, is bounded above by a constant depending on \( a \) and \( b \). So we can forget \( g'_1 \) and continue the construction of our \( \frac{1}{a} \)-stair relative to \( C \) at the point where the approximation of \( g'_1 \) leaves the \( r \)-neighborhood of \( C_k \). We eventually exhaust \( g \) and obtain a \( \frac{1}{a} \)-stair relative to \( C \). □

8.5. **Proof of Proposition 3.10.** Let \( g \) and \( C \) be as given by this proposition. Assume that some subchain \( g' \) of \( g \) leaves and then reenters the horizontal \( D_{8,7} \)-neighborhood of \( C \). Assume that \( g' \) is not contained in the telescopic \( C_{8,7}(D_{8,7}, a, b) \)-neighborhood of \( C \). We set \( C_{8,7} := C_{8,7}(D_{8,7}, a, b) \) and \( E_{8,7} := E_{8,7}(D_{8,7}, a, b) \).

Suppose for the moment that the vertical segments in \( g' \) have vertical length greater than \( 3(C_{2,19} + D_{7,1}) \). Then Lemma 8.7 gives \( G \), a \( E_{8,7} \)-stair relative to \( C \) with \( d_{\text{tel}}^H(g', G) \leq C_{8,7} \). From Lemma 8.6, \( G \) does not leave the vertical \( C_{8,6}(E_{8,7}) \)-neighborhood of the stratum containing the initial point of \( G \). Therefore, by setting \( V(a, b) = C_{8,6}(E_{8,7}) + C_{8,7} \), \( g' \) does not leave the vertical \( V(a, b) \)-neighborhood of this stratum. From Lemma 8.3, \( g' \) lies in the telescopic \( C_{8,3}(V(a, b), a, b) \)-neighborhood of \( C \).

It remains to consider the case where the vertical segments in \( g' \) are not sufficiently large enough. Let \( s \) be a vertical segment in \( g \) with \( |s|_{\text{vert}} < X := 3(C_{2,19} + D_{7,1}) \).

(§) Thanks to the assumption that all the attaching-maps of the tree of hyperbolic spaces are quasi isometries, \( s \) is contained in a vertical segment \( s' \) of vertical length greater than \( X \). We modify \( g' \) by sliding, along \( s' \), a horizontal geodesic in \( g' \) incident to \( s \) until getting a vertical segment with vertical length \( X \). This yields a new telescopic \((a', b')\)-quasi geodesic in a bounded neighborhood of \( g \), where the constants \( a', b' \) only depend on \( a, b \) and on the constants of quasi isometry. After finitely many such moves, we obtain a quasi geodesic as desired, and we are done. Since the vertical distance between two strata is uniformly bounded away from zero, after finitely many such substitutions, we eventually get a quasi geodesic, in a bounded neighborhood of \( g \), which satisfies the assumptions required by Lemma 8.7. This completes the proof of Proposition 3.10. □

8.6. **Adaptation to generalized corridors.** The only problem is to get a telescopic chain with maximal vertical segments sufficiently large enough. We start from the sentence marked by a (§) in the preceding subsection. If \( s \) is not contained in a vertical segment \( s' \) of vertical length greater than \( X \), we obtain a vertical segment \( s \) from \( b_i \) to \( a_{i+1} \) satisfying the following properties (we still denote by \( g' \) the \((a', b')\)-quasi geodesic eventually obtained, we denote by \( s_0 \) the vertical segment of \( g' \) ending at \( a_i \) and by \( s_i \) the one starting at \( b_{i+1} \)):

(a) there is no vertical segment starting at \( a_i \) (resp. at \( a_{i+1} \)) over the edge \( \pi(s) \) (resp. over \( \pi(s_i) \));

(b) there is no vertical segment ending at \( b_i \) over \( \pi(s_0) \).

Consider horizontal geodesics \( \alpha_i = [a_i, P_{C}^{\text{hor}}(a_i)] \), \( \beta_i = [b_i, P_{C}^{\text{hor}}(b_i)] \), \( \alpha_{i+1} = [a_{i+1}, P_{C}^{\text{hor}}(a_{i+1})] \) and \( \beta_{i+1} = [b_{i+1}, P_{C}^{\text{hor}}(b_{i+1})] \). By the \( \delta \)-hyperbolicity of the strata, there is \( a'_i \in [a_i, b_i] \cap N_{\delta}^{28}(\alpha_i \cup \beta_i) \) and \( b'_i \in [a_{i+1}, b_{i+1}] \cap N_{\delta}^{28}(\alpha_{i+1} \cup \beta_{i+1}) \). Because the strata are quasi isometrically embedded one into each other, we get two points \( a'_i, b'_i \) which satisfy:

(A) they are \( Y \)-close (with respect to the horizontal distance) respectively to \( a'_i \) and \( b'_i \), where the constant \( Y \) only depends on \( \delta \) and on the constants of quasi isometry.
(B) there is a \( v \)-vertical segment from \( a''_i \) to \( b''_i \) which is contained in a larger \( v \)-vertical segment going over \( \pi(s_0) \) and \( \pi(s_1) \).

We modify \( g' \) by going from \( a_i \) to \( a''_i \) then to \( b''_i \) and eventually end at \( b_i+1 \). The resulting chain is a \( (a'', b'') \)-quasi geodesic, where the constants \( a'', b'' \) only depends on \( \delta \) and on the constants of quasi isometry. Moreover this new chain is in a bounded neighborhood of \( g' \). Thanks to Item (B), we can modify it by enlarging the vertical segment from \( a''_i \) to \( b''_i \). The conclusion in then the same as in the preceding subsection. \( \square \)

8.7. Proof of Proposition 6.14. The arguments are similar to those exposed for proving the quasi convexity of the corridors. We give here only a sketch of the proof. The horizontal deviation of an exceptional orbit with respect to \( C \) depends linearly on the vertical variation of the orbit (Lemma 6.6). Thus, if a sufficiently large segment of the orbit remains outside a sufficiently large horizontal neighborhood of \( C \), the exponential separation implies that the horizontal distance between the orbit and \( C \) exponentially increases with the vertical length of the orbit. Assume now that the exceptional orbit considered is followed by another one. The strong exponential separation gives the same consequence: this second exceptional orbit does not go back to \( C \) and the horizontal distance with respect to \( C \) exponentially increases with its vertical length, as soon as this length is sufficiently large enough. Here the arguments are similar to those used for proving Lemmas 8.5 and 8.6. Finally, if the exceptional orbit is followed by a quasi geodesic in \( \hat{X} \), then the approximation by a stair as was done before, yields the same conclusion. \( \square \)

References


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